Non uniform (hyper/multi)coherence spaces

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Abstract

In (hyper)coherence semantics, proofs/terms are cliques in (hyper)graphs. Intuitively, vertices represent results of computations and the edge relation witnesses the ability of being assembled into a same piece of data or a same (strongly) stable function, at arrow types.

In (hyper)coherence semantics, the argument of a (strongly) stable functional is always a (strongly) stable function. As a consequence, comparatively to the relational semantics, where there is no edge relation, some vertices are missing. Recovering these vertices is essential for the purpose of reconstructing proofs/terms from their interpretations. It shall also be useful for the comparison with other semantics, like game semantics.

In [BE01], Bucciarelli and Ehrhard introduced a so called *non uniform coherence space semantics* where no vertex is missing. By constructing the co-free exponential we set a new version of this last semantics, together with non uniform versions of hypercoherences and multicoherences, a new semantics where an edge is a finite multiset. Thanks to the co-free construction, these non uniform semantics are deterministic in the sense that the intersection of a clique and of an anti-clique contains at most one vertex, a result of interaction, and extensionally collapse onto the corresponding uniform semantics.

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Notations. In this paper, multiset will always means *finite* multiset. We use the notation $[\ ]$ for multisets while the notation $\{\ \}$ is, as usual, for sets. The pairwise union of multisets is denoted by a $+$ sign and following this notation the generalised union is denoted by a $\sum$ sign. The neutral element for this operation, the empty multiset, is denoted by $[]$. If $k \in \mathbb{N}$, $k[a]$ denotes the multiset $\sum_1^k [a]$. If $[a_i \mid i \in I]$ is a multiset, its support is the set $\{a_i \mid i \in I\}$. The cardinality $\sharp[a_i \mid i \in I]$ of a multiset $[a_i \mid i \in I]$ is the cardinality $\sharp I$ of the set I. If m is a multiset we denote by $\mathrm{supp}(m)$ its support. The disjoint sum operation on sets is defined by setting $A + B = \{1\} \times A \cup \{0\} \times B$. The categorical composition is denoted by $\fatsemi$.

1 Introduction

1.1 Strong stability and hypercoherences

Strong stability has been introduced by Bucciarelli and Ehrhard in [BE94] for the purpose of giving a purely "extensional" definition of sequentiality at all types, that is, a description of sequential computations which does not involve the atomic description of each step of interaction of an agent (function, term) with its environment (argument, or more generally, context), as game semantics do. The results obtained by Ehrhard in [Ehr99] and later proved again by Longley [Lon02], Van Oosten [vO97] and Melliès [Mel05], with different methods, showed that indeed, strong stability corresponds to sequentiality at all types. Ehrhard established that the strongly stable semantics is the extensional collapse of the sequential algorithm semantics designed in the late 70's by Berry and Curien [BC82]. Unlike the continuous or stable interpretations of PCF, the sequential algorithm interpretation (which is now better understood as a deterministic game semantics) is very "operational" in nature: Cartwright, Curien and Felleisen showed in [CCF94] that sequential algorithms are fully abstract (and fully complete) for the extension of PCF by a *catch and throw* mechanism. In [Lon02], Longley advocates the claim that there exists a canonical notion of "sequential" functionals of all types which coincides with the hierarchy of strongly stable functions.

This comparison of the strongly stable semantics with more operational interpretations has been made possible only by the discovery of *hypercoherences* by Ehrhard [Ehr93]. Moreover, the introduction of these objects simplified the presentation of the strongly stable semantics and provided a strongly stable interpretation of (second order) linear logic. A hypercoherence is very similar to a coherence space [Gir87] and consists of a set, the *web*, together with a coherence relation on this web. However, in a hypercoherence, the coherence relation is not a binary relation, but a set of finite subsets of the web containing all singletons (these sets are said to be coherent). An

"element" of a hypercoherence X is then a *clique* of X, that is, a subset of the web of X which has the property that all its finite and non-empty subsets are coherent.

Hypercoherences are a semantics of linear logic, so they provide an interpretation of intuitionistic implication which is of the shape $X \Rightarrow Y = (!X) \multimap Y$ where "$\multimap$" is a linear implication and ! is a so called "exponential". The basic operational intuition behind this decomposition is as follows: a linear map represents a program which uses its argument exactly once, and an element of $!X$ is obtained essentially by taking an element of X and making it available as many times as required.

The purely relational semantics is maybe the simplest semantics of linear logic. In this semantics formulæ are sets and proofs are relations. The constructions of the relational semantics underly both the coherence space semantics and the hypercoherence semantics. Barreiro and Ehrhard traced back the introduction of the relational semantics as induced by an unpublished remark from van de Wiele about the co-freeness of the exponentials in coherence semantics.

The hypercoherence semantics is said to be static as opposed to games semantics which involve a direct representation of the dynamics of computation. In game semantics, time is explicit: such semantics interpret terms by focusing on the history of an atomic interaction between a player (the program implemented by the term) and an opponent (the environment). For instance an interaction inside a function type $A \to B$ is an interleaving of an interaction querying a piece of A data and an interaction producing a piece of B data.

There is no such reference to time in hypercoherences. For instance, the web of a linear function space is the Cartesian product of the webs.

However the strong relation of hypercoherences with sequentiality means that the semantics carries an *implicit* representation of time.

In [Mel05], Melliés investigate the game theoretic counterpart of this implicit representation of time, by introducing sequential games in which the coherence relation can be expressed in game terms.

In a complementary direction, we used an unfolding of hypercoherences introduced by Ehrhard in [Ehr00], to uncover the game structures of hypercoherences [Bou04]. For the reverse direction, the idea was to project directly usual games onto hypercoherences by mapping history of interaction onto results of interaction. But some history of interaction were not mapped anywhere in the hypercoherence semantics. Indeed, not only the representations of interaction differ between games and hypercoherences but these two kinds of semantics do not agree on what are the possible interaction between terms. More precisely, one can circumvent the problem by projecting onto the relational semantics rather than hypercoherences [Bou05]. In the relational semantics the representation of an interaction is the same as in hypercoherences but their is less (or no) assumption in the relational semantics about the possibilities of interactions.

1.2 Uniformity

The relational semantics almost consists of the part of the hypercoherence semantics dealing with webs, except that in hypercoherences the web of the exponentials depends on the coherence relations. To be precise the web of $!A$ in the relational semantics is the set of multisets of elements of the web of A while, in hypercoherences the web of $!A$ contains only multisets which supports are clique.

The dependence of webs on coherence is what is called uniformity of the exponentials. This terminology, mainly used by Ehrhard and Girard, comes from the fact that in such semantics the context of an agent behaves uniformly, that is: as if this context is produced by a single agent. The hypercoherence interpretation of a term omits points relatively to its relational interpretation and so the hypercoherence semantics loses information about some parts of the term. The same holds for the coherence semantics.

Lets take an example (very standard). The relational semantics of the simply typed term

$$\lambda b^{\mathbf{bool}}.\, \mathtt{if}\ b\ \mathtt{then}\ (\mathtt{if}\ b\ \mathtt{then}\ \mathtt{v}\ \mathtt{else}\ \mathtt{v})\ \mathtt{else}\ (\mathtt{if}\ b\ \mathtt{then}\ \mathtt{f}\ \mathtt{else}\ \mathtt{f}). \tag{1}$$

(where $\mathtt{v}$ stands for true and $\mathtt{f}$ for false) is the relation

$$\{([\mathtt{v},\mathtt{v}],\mathtt{v}),([\mathtt{f},\mathtt{v}],\mathtt{v}),([\mathtt{f},\mathtt{v}],\mathtt{f}),([\mathtt{f},\mathtt{f}],\mathtt{f})\}$$

but its hypercoherence semantics is just $\{([\mathtt{v},\mathtt{v}],\mathtt{v}),([\mathtt{f},\mathtt{f}],\mathtt{f})\}$.

The hypercoherence semantics of the term trusts its environment and makes the assumption that the boolean b has one fixed value during the time of the computation. Of course, this is fair from an interactive viewpoint since the environment complies with coherence conditions as programs do. But for the purpose of reconstructing terms from their semantics, some information is missing. Our example is very simple, but it is easy to imagine terms where the part missed by the uniform interpretation contains big sub-routines rather than constants.

Intuitively, one can think of uniformity as a technique to remove a particular kind of dead code, as in the example above. However, it is worth to remark that what is lost by uniformity in coherence spaces and hypercoherences can hardly, in general and especially at functional types, be match with well-identified pieces of syntax (sub-terms or sub-proofs). First these semantics are not fully complete, so (a part of) a semantical agent not always corresponds to (a part of) a syntactical agent. Second the uniformity restriction not only takes into account accessibility of branches of code but also, in a more subtle manner, reachability constraints, in particular on the copying discipline of pieces of data (see Example 2).

Non uniform static semantics will interpret terms exactly as the relational semantics does. This will allow us to *combine* semantics in order to take advantage of their different features. For instance, we can define a semantics where proofs will be cliques both in the non uniform coherence space semantics, in the non uniform hypercoherence semantics and in the non uniform bipartite semantics we present in this paper.

The uniformity/non uniformity issue in static semantics is to be related with games where some uniformity conditions were originally designed for the exponential type : interactions in $!A$ are *deterministic* (in the sense of games) interleaving of interactions of A, see [AJM94]. Recent works in the game semantics area are more permissive: such conditions (games determinism) on the semantics of types are postponed to conditions on the semantics of terms.

1.3 The former (hyper)coherence semantics

Providing coherence space or hypercoherence semantics with non uniform exponentials is not a trivial job. One has to design a semantics where for instance, one point of the web shall be incoherent with himself. This must be the case for the point $[\mathtt{v},\mathtt{f}]$ since the valid term above maps it to an incoherent piece of data $\{\mathtt{v},\mathtt{f}\}$. The situation where two different points are coherent and incoherent at the same time may also arise. In coherence spaces this will mean the semantics does not enjoy *determinism* —we come back with this latter.

However, the main difficulty lies in defining the interpretation of the exponentials.

A. Bucciarelli and T. Ehrhard have designed a general tool for producing non uniform semantics see [BE01]. As observed by J.-Y. Girard in [Gir96], to be closer to full completeness for linear logic, the coherence spaces semantics can be enriched by indexing each clique on a monoid. To make the story short, by doing this and thanks to a clever handling of *indexes* (locations), A. Bucciarelli and T. Ehrhard obtained that when this monoid comes with a *phase space* structure of a certain sort (actually, a symmetric phase space which is a truth-value semantics of an *indexed* linear logic) this leads to a denotational semantics of linear logic. For details see [BE00] and [BE01]. This leaves us with, potentially, an infinity of denotational semantics of linear logic. A. Bruasse-Bac has studied many of them in her PhD. thesis [Bru01] among which there is one rejecting the *Mix* rule. A quite simple phase space produces a former version of non uniform coherence semantics. According to a suggestion of Ehrhard and Bucciarelli, by generalizing the construction to all arities one obtains non uniformity for something sounding like non uniform hypercoherence semantics. But:

1. each of these non uniform semantics badly relates with their usual (*e.g.* uniform) versions;
2. neither the coherence, nor the hypercoherence non uniform semantics are deterministic;

3. Furthermore, we observed that the former non uniform hypercoherence semantics misses one important feature of the usual hypercoherence semantics : not all the finite cliques of type $\mathbf{bool}^n \to \mathbf{bool}$ are *sub-definable* (*i.e.* included in the interpretation of a term).

4. The former non uniform coherence semantics is also a little bit puzzling. For instance, in $!\mathbf{bool}$, for any $p, q \in \mathbb{N}$ such that $p + q > 1$, the points $p[\mathtt{v}]$ and $q[\mathtt{v}]$ are coherent and incoherent at the same time (while, once $p \neq q$, they are strictly coherent in usual coherence spaces). So, one can find in the former non uniform semantics a semantical agent mixing the term (1), querying two times its argument, and the term $\lambda b.\mathtt{v}$, which does not use its argument. (This is not the case in the former hypercoherence semantics).

1.4 Contribution

The present work is an extended version of our previous communication at the CTCS conference [Bou03]. Some parts were also presented in our PhD.

Our starting point was the former non uniform (hyper)coherence semantics and we mainly focused on hypercoherences. We observed that:

1. contrarily to what happens in the usual hypercoherence semantics, there is at least one clique f of type $\mathbf{bool}^n \to \mathbf{bool}$ which is not included in the interpretation of any term. The clique f is a variant of one originally designed by Berry to reveal the same failure in coherence spaces[1]. What is important to notice is that f does not contain points related to non uniformity, such as $[\mathtt{v}, \mathtt{f}]$. Hence the set f can be presented to the usual (multiset based) hypercoherence semantics which successfully refutes it.

2. Many others definitions of the interpretation of exponentials are possible.

Among all the possibilities for the interpretation of exponentials we found the *co-free* exponential (think of it as to be an *infinite* tensor product). This has led to a more satisfactory setting both for coherence spaces and hypercoherences. We also introduced a new coherence like semantics, multicoherences.

1. The co-free exponential is maximal in a sense we make precise in Corollary 1 and which basically means that any clique of type $\mathbf{bool}^n \to \mathbf{bool}$ would also be a clique with others variants of the exponentials. The bad news is that there still exists a clique of type $\mathbf{bool}^n \to \mathbf{bool}$ which is not included in the interpretation of any (sequential) term. But, in hypercoherences (and multicoherences), such cliques necessarily contain points coming from non uniformity. So, this phenomenon is now constrained to the non uniform web and disappears when restricting to the uniform web.

2. Our non uniform semantics are *deterministic* in the sense that the intersection of a clique (let say of type A) and an anti-clique (a clique of dual type $A^\perp$) contains at most one point, like in the uniform semantics. We call *results of interactions*, the points at the intersection of a clique and an anti-clique. Not all points can be results of interactions. For instance $[\mathtt{v}, \mathtt{f}] \in !\mathbf{bool}$ is not, since it is strictly incoherent with itself (and so does not belong to any clique of type $!\mathbf{bool}$).

3. For both coherence spaces, hypercoherences and multicoherences, we derive the usual uniform interpretation of types and terms by a straight set restriction of the web to results of interactions (both at the the level of types and at the level of proofs). More precisely, the restriction gives the multiset based uniform interpretation (the web of $!A$ is made of multisets which supports are cliques). Note that is was already known that the restriction of the relational interpretation of a proof to the web of the coherence space interpretation of a formula is the coherence space interpretation of the proof [Tor00]. The novelty introduced here is mainly that the uniform web is characterized by the coherence relation.

[1] Berry's example is often called the *Gustave's function* which is named after a private joke about the huge number of french scientists whose first name is Gérard, among which Berry.

Table 1: A summary

Semantics : **Coherence :** **Category :**	relational empty **Rel**	coherence spaces pairs $\mathbf{NCoh}_{\{2\}}$	hypercoherences sets **NHc**	multicoherences multisets $\mathbf{NCoh}_{\mathbb{N}\setminus\{0,1\}}$	bipartite singletons **Bip**
Web of the exponentials (variants)					
multisets (co-free)	yes	new version	new	new	new
multisets uniform	no	yes	yes	new	new
sets uniform	no	yes	yes	new	new

4. In each case, the non uniform and the multiset based uniform semantics are extensionally equivalent : they have the same extensional collapse. As already known [BE97, Mel04] for the multiset based semantics, this extensional collapse is the set based uniform semantics.

5. The existence of deterministic non uniform semantics implies an unexpected property. Consider an extension of linear logic with new rules, typically the daemon of Girard's Ludics [Gir01], for which the semantics is still valid and such that A and $A^{\perp}$ are both provable. Then a cut between a proof of A and a proof of $A^{\perp}$ induces an interaction which involves at most one point in the relational semantics. This unexpected result would have been hard to prove without introducing non uniform (hyper/multi)coherence semantics. It may prove useful with other semantics based on the relational semantics. For instance, in Ehrhard's finiteness spaces [Ehr05], points of interaction are equipped with multiplicities. Since there is at most one point for each interaction, one can use its multiplicity to do some quantitative analysis of proofs' interaction.

6. The multicoherence semantics aroses as the general case in our approach of non uniformity. In fact, we derive the non uniform hypercoherence semantics from multicoherences. The difference with hypercoherences is that the coherence relation is made of multisets rather than sets. As for uniform hypercoherences, at first order simple types, each finite clique in the uniform multicoherence semantics is sub-definable. But contrarily to hypercoherences, each clique of the multicoherence semantics is a clique in coherence spaces. At a functional type, there exists sets which are cliques in the hypercoherence semantics but which are not cliques in the multicoherence semantics (even in the set based uniform case). Since the set based uniform multicoherence semantics is extensional, there is at least two extensionally different semantics of higher order: multicoherences and hypercoherences.

In Table 1, we summarize the principal variants of coherence based semantics, with our new ones. Note that there are two axis where one can chose between sets and multisets: either for the coherence relation (the power of the coherence) or for the web of the exponentials. A third one, which do not appear in that paper, is the shift from sets to multisets for cliques, as in finiteness spaces [Ehr05].

The (non uniform) bipartite semantics of linear logic we present in Section 3 comes originally from a simple remark about the relational semantics. This remark states that one can set polarities (positive/negative) on points of the web in such a manner that the orthogonal exchanges polarities and that every proof is interpreted by a set of positive points. This gives a kind of coherence spaces semantics where the coherence relation is of arity one. The semantics one obtains is non uniform. We discovered that this bipartite semantics also admits a uniform version where every proof of a *why not* formula is interpreted by the empty set. Besides this radical lapse of memory of points in proofs interpretation, the uniform bipartite semantics (as the non uniform bipartite semantics) is equivalent to the relational semantics on simple types (proofs and types interpretations are the same and these three semantics are extensionally equivalent).

In a recent work [Pag06] Michele Pagani showed that there is a syntactical counterpart, *visible acyclicity*, to non uniform coherence : our non uniform coherence space semantics corresponds to a relaxation of the correctness criterion of linear logic proof-nets, (a graphical presentation of proofs). Finding a similar correspondence for non uniform hypercoherences or multicoherences would certainly be interesting.

1.5 Contents

The next section (Section 2) is devoted to recalling briefly some useful definitions and properties we deal with. The only novelty is the introduction of a convenient framework, P-cohérence spaces, to deal with various static semantics.

In Section 3, we present the bipartite semantics and its uniform version, and we compare them. We stress that the bipartite semantics are just here as a peculiar example of uniform/non uniform setting, but do not give an example of the general uniform/non uniform construction we use further.

Section 4 and Section 5 form the core of the paper, where we study the exponential coming from indexed linear logic, and develop their co-free version. This part is an extended version of our conference paper on non-uniform hypercoherences [Bou03]. In this part, we present our results by mostly following the chronological order of their discovery.

We start with the presentation of K-cohérent spaces, a denotational semantics coming from indexed linear logic and aimed to be a generalization of (hyper)coherence spaces. The semantics is parametrized by a set K which encompasses (a kind of) coherence spaces $K = \{2\}$ and (a kind of) hypercoherences $K = \mathbb{N}\backslash\{0, 1\}$. We then point out the definability problem at first order.

We further introduce the co-free exponentials (Subsection 4.4). We show that the co-free exponentials provide the best solution one can expect (Corollary 1). We show that there are still some definibility problems (Corollary 2) but only due to non-uniformity.

We then establish (Subsection 4.5) that the K-cohérent semantics equipped with the new exponentials gives a deterministic semantics. In fact, the semantics satisfies a property stronger than just determinism. This allows the introduction of a web restriction operation, the neutral functor (Subsection 5.1) giving the uniform version of the K-cohérent semantics (Subsection 5.2).

The uniform semantics obtained for $K = \mathbb{N}\backslash\{0, 1\}$ is not hypercoherences but a new one, multicoherences (Subsection 5.3). Hypercoherences and non-uniform hypercoherences are obtained by an operation forgetting multiplicities in the coherence relation (Subsection 5.4).

We close this part by a study which relates the extensional collapses of the various semantics (Subsection 5.5).

In the concluding section, we adopt an interactive viewpoint à la Girard to discuss the implications of the existence of deterministic non-uniform semantics (Subsection 6.1).

2 Preliminaries

2.1 Extensional collapse

Extensional partial equivalence relations were first introduced by Kreisel in the fifties to deal with higher order partial recursive functions. An extensional PER is meant to relate two algorithms when they implement the same function. Higher order is responsible for the partiality of the equivalence relation.

Simple types are types of the simply typed lambda calculus enriched with basis types in order to form a type system for PCF. They are given by the following grammar:

$$\sigma, \tau := \iota \mid \sigma \to \tau \quad \text{(simple types)}$$

where ι stands for basis types, typically a boolean type **bool** or/and a natural number type **nat**. A product type $\sigma \times \sigma'$ can also be introduced but we won't bother with this type constructor since it can be obtained by curryfication.

Let $\mathcal{M}$ be a categorical semantics of linear logic, let $\mathcal{M}(A)$ denotes the interpretation of a type A and let us call semantical agents of type A the elements of the semantics used to interpret proofs of A, that is morphisms from 1 to $\mathcal{M}(A)$. Suppose an interpretation of basis types is given in $\mathcal{M}$ (usually **bool** is interpreted as the space $1 \oplus 1$ and **nat** is interpreted as an ω-infinite *plus* of 1). Then we extend this intepretation of basis types to all simple types by setting $\mathcal{M}(\sigma \to \tau) = !\mathcal{M}(\sigma) \multimap \mathcal{M}(\tau)$. So, the function type corresponds to the object of morphisms in

the co-Kleisli category. If f and x are semantical agents of respective types $\sigma \to \tau$ and σ then we *apply* f to x by composing in the co-Kleisli category to form a semantical agent $f(x)$ of type τ. For each simple type σ, an extensional PER $\sim_\sigma$ is defined on semantical agents of type σ by chosing the equality on basis types and by setting:

$$f \sim_{\sigma\to\tau} g \text{ iff if } x \sim_\sigma y \text{ then } f(x) \sim_\tau g(y).$$

The *extensional collapse* of the semantics is the set of quotients by extensional PERs of the interpretations of simples types equipped with the following notion of application. If $\overline{f}$ is a class of semantical agents of type $\sigma \to \tau$ (functions) and if $\overline{x}$ is a class of semantical agents of type τ (arguments) the application of $\overline{f}$ to $\overline{x}$ is defined by setting $\overline{f}(\overline{x}) = \overline{f(x)}$.

2.2 Power coherence spaces

We introduce a general notion which will provide us with a very convenient language for describing the various semantics we deal with. A *power* is simply a functor from the category of sets and inclusions to itself. Typical powers relevant to our purpose are:

- The empty power defined by $E \mapsto \emptyset$. This power will simply be denoted $\emptyset$. It can be used to present the relational semantics in terms of power coherence spaces;
- The identity power, id, which will be used for dealing with the bipartite relational semantics of Section 3;
- The non-empty finite sets power $\mathcal{P}^*_{\text{fin}}$ which maps each set to the set of its finite non-empty subsets. The power $\mathcal{P}^*_{\text{fin}}$ will be used for dealing with hypercoherences;
- Given a subset K of $\mathbb{N} \setminus \{0, 1\}$, the power $\mathcal{M}_K$ which maps a set E to the set of all finite multisets over E whose cardinality belongs to K. The power $\mathcal{M}_{\{2\}}$ will be used for dealing with coherence spaces. The choice of this power follows the suggestion made at the end of [BE01] for the purpose of building non uniform coherence or hypercoherence like semantics.

Définition 1. Let P be a power. A *P-cohérence space* X is a triple $(|X|, \mathrel{\overset{\frown}{\smile}}_X, \asymp_X)$ where $|X|$ is an at most countable set, the *web* of X, and where $\mathrel{\overset{\frown}{\smile}}_X$ and $\asymp_X$ are subsets of $P(|X|)$ such that $\mathrel{\overset{\frown}{\smile}}_X \cup \asymp_X = P(|X|)$. The set $\mathrel{\overset{\frown}{\smile}}_X$ is called the *coherence* and the set $\asymp_X$ is called the *incoherence*. The intersection of $\mathrel{\overset{\frown}{\smile}}_X$ and $\asymp_X$ is called the *neutrality*. Notation: N_X. The strict coherence $\frown_X$ of X is the complementary set of $\asymp_X$ with respect to $P(|X|)$ and the strict incoherence $\smile_X$ is the complementary of $\mathrel{\overset{\frown}{\smile}}_X$.

Clearly, one can define a P-cohérence space X by specifying two well chosen sets among $\mathrel{\overset{\frown}{\smile}}_X$, $\asymp_X$, N_X, $\frown_X$ and $\smile_X$ subject to obvious constraints (for instance, one must have $\mathsf{N}_X \subseteq \mathrel{\overset{\frown}{\smile}}_X$, $\smile_X \cap \frown_X = \emptyset \ldots$).

Définition 2. The orthogonal, $X^\perp$, of a P-cohérence space $X = (|X|, \mathrel{\overset{\frown}{\smile}}_X, \asymp_X)$ is the P-cohérence space $(|X|, \asymp_X, \mathrel{\overset{\frown}{\smile}}_X)$. (Orthogonality exchanges coherence and incoherence).

Définition 3. Let X be a P-cohérence space. A *clique* of X is a subset x of $|X|$ such that $P(x) \subseteq \mathrel{\overset{\frown}{\smile}}_X$. We denote by $\mathrm{Cl}(X)$ the set of all cliques of X. An *anti-clique* of X is a clique of $X^\perp$. If for each clique x and each anti-clique y the intersection of x and y contains at most one element then P-cohérence space X is *deterministic*.

Définition 4. A P-cohérence space X is *reflexive* if neutrality corresponds to equality in the sense that:

$$\mathsf{N}_X = \cup_{a\in|X|} P(\{a\}). \tag{2}$$

A P-cohérence space X is *weakly reflexive* if

$$\mathsf{N}_X \subseteq \cup_{a\in|X|} P(\{a\}). \tag{3}$$

One can define a reflexive P-cohérence space by specifying only $\mathrel{\overset{\frown}{\smile}}_X$ (or $\asymp_X$, or $\frown_X$, or $\smile_X$).

Proposition 1. *If the power is strictly monotone and preserves disjointness of sets, then weak reflexivity implies determinism.*

Proof. Lets take a P-cohérence space X, and a clique and an anti-clique with at least two points, a and b at their intersection. Then $\{a, b\}$ is both a clique and an anti-clique. Thus $P(\{a, b\}) \subseteq \mathrel{\overset{\frown}{\smile}}_X \cap \asymp_X$. By strict monotonicity and preservation of disjointness $P(\{a, b\}) \not\subset \cup_{c \in |X|} P(\{c\})$ which contradicts weak reflexivity. $\square$

But weak reflexivity is in general stronger than just determinism. For instance, in a $\mathcal{P}^*_{\text{fin}}$-space X one can find a set $\{a, b, c\} \in \mathcal{P}^*_{\text{fin}}(|X|)$ which is both coherent and incoherent (so the space is not weakly reflexive) and still have determinism (take for instance $\frown_X = \{\{a, b\}, \{a, c\}, \{b, c\}\}$ and $\smile_X = \emptyset$).

Weak reflexivity and determinism are equivalent in $\mathcal{M}_{\{2\}}$-spaces.

2.3 Relational semantics

We recall briefly the interpretation of linear logic in the category **Rel** of sets and relations.

Let us recall that the composition is given by:

$$f \mathbin{;} g = \{(a, c) \mid \exists b, (a, b) \in f \text{ et } (b, c) \in g\}$$

and that identities are given by:

$$\mathrm{id}_X = \{(a, a) \mid a \in |X|\}.$$

Formulae. A formula A is interpreted by a set $|A|$ defined inductively as follows: $|0| = |\top| = \emptyset$, $|1| = |\bot| = \{*\}$, $|A^\perp| = |A|$, $|A \oplus B| = |A \& B| = |A| + |B|$, $|A \otimes B| = |A ⅋ B| = |A| \times |B|$ and $|!A| = |?A| = \mathcal{M}_{\text{fin}}(|A|)$ where $\mathcal{M}_{\text{fin}}(E)$ is the set of finite multisets on E.

Sequents. We use the right-sided presentation of the linear logic sequent calculus. Up to associativity and commutativity of the Cartesian product, the "comma" of sequents is safely interpreted as a *par i.e.* by setting $|\vdash A_1, \ldots, A_n| = |A_1 ⅋ \ldots ⅋ A_n|$ which is equal to $|A_1| \times \ldots \times |A_n|$.

Proofs. The interpretation of a proof of a sequent $\vdash \Gamma$ is a subset of $|\vdash \Gamma|$ defined inductively on the proof, by cases on the last rule, as shown below.

It is well-known that this interpretation is a denotational semantics of linear logic (that is: two proofs of a given sequent have the same interpretation as soon as they are equivalent up to cut-elimination).

Identity group

$$\frac{}{\vdash A, A^\perp : \{(a, a) \mid a \in |A|\}}\,(ax.) \qquad \frac{\vdash \Gamma, A : f \qquad \vdash \Delta, A^\perp : g}{\vdash \Gamma, \Delta : \{(\gamma, \delta) \mid \exists a, (\gamma, a) \in f \wedge (\delta, a) \in g\}}\,(cut)$$

Additives

$$\frac{}{\vdash \Gamma, \top : \emptyset}\,(top) \qquad \frac{\vdash \Gamma, A : f \qquad \vdash \Gamma, B : g}{\vdash \Gamma, A \& B : f \uplus g}\,(avec) \qquad \frac{\vdash \Gamma, A : f}{\vdash \Gamma, A \oplus B : f}\,(plus)$$

Multiplicatives

$$\frac{\vdash \Gamma : f}{\vdash\!\vdash \Gamma, \bot : f \times \{*_\bot\}}\ (bot) \qquad \frac{}{\vdash\!\vdash 1 : \{*_1\}}\ (un)$$

$$\frac{\vdash \Gamma, A, B : f}{\vdash\!\vdash \Gamma, A \parr B : f}\ (par) \qquad \frac{\vdash \Gamma, A : f \qquad \vdash \Delta, B : g}{\vdash\!\vdash \Gamma, \Delta, A \otimes B : \{(\gamma, \delta, (a, b)) \mid (\gamma, a) \in f, (\delta, b) \in g\}}\ (tens.)$$

Exponentials

$$\frac{\vdash ?A_1, \ldots, ?A_n, A : f}{\vdash ?A_1, \ldots, ?A_n, !A : f^\dagger}\ (prom.) \qquad \frac{\vdash \Gamma, ?A, ?A : f}{\vdash \Gamma, ?A : \{(\gamma, \mu_1 + \mu_2) \mid (\gamma, \mu_1, \mu_2) \in f\}}\ (cont.)$$

$$\frac{\vdash \Gamma : f}{\vdash \Gamma, ?A : \{(\gamma, []) \mid (\gamma) \in f\}}\ (aff.) \qquad \frac{\vdash \Gamma, A : f}{\vdash \Gamma, ?A : \{(\gamma, [a]) \mid (\gamma, a) \in f\}}\ (der.)$$

Where $f^\dagger$ is equal to :

$$\{(\sum_{j \in J} \mu_1^j, \ldots, \sum_{j \in J} \mu_n^j, [a_j \mid j \in J]) \mid [(\mu_1^j, \ldots, \mu_n^j, a_j) \mid j \in J] \in \mathcal{M}_{\mathrm{fin}}(f)\}.$$

The relational semantics is actually a categorical semantics of linear logic, though we shall not recall its categorical structure in details. The new Seely categorical semantics axiomatic [Bie95] is appropriate for dealing with the relational semantics and we will use this axiomatic for further semantics. Exponentials are given by a comonad structure $(!, \mathrm{der}, \mathrm{dig})$. We just recall this structure. The endofunctor $!$ of **Rel** is defined by $!E = \mathcal{M}_{\mathrm{fin}}(E)$ and

$$!f = \{([a_i \mid i \in I], [b_i \mid i \in I]) \mid [(a_i, b_i) \mid i \in I] \in \mathcal{M}_{\mathrm{fin}}(f)\}.$$

The natural transformations $\mathrm{der} : ! \dot{\to} \mathrm{id}$ and $\mathrm{dig} : ! \dot{\to} !!$ are defined by setting:

$$\begin{aligned} \mathrm{der}_E &= \{([a], a) \mid a \in E\} \\ \mathrm{dig}_E &= \{(\sum_{i \in I} \mu_i, [\mu_i \mid i \in I]) \mid [\mu_i \mid i \in I] \in !!E\}. \end{aligned}$$

2.4 Coherence spaces

We briefly recall the coherence spaces semantics of linear logic [Gir87].

A coherence space is a reflexive $\mathcal{M}_{\{2\}}$-coherence space.

We define directly the connectives of linear logic on coherence spaces (rather than defining by induction the interpretation of formulae). The web of multiplicatives and additive is the same as in the relational semantics. Coherence is defined as follows. One has $\frown_{X \oplus Y} = \frown_X \uplus \frown_Y$ and $[(a, b), (a', b')] \in \frown_{X \otimes Y}$ iff $[a, a'] \in \frown_X$ and $[b, b'] \in \frown_Y$.

Morphisms from X to Y in the corresponding (linear) category are just cliques of $X \multimap Y = X^\perp \parr Y$.

The interpretation of proofs in coherence spaces only differs from the relational semantics on exponential rules. Let us recall that the interpretation of a proof is just a subset of the web of the interpretation of the conclusion sequent and that this is a corollary of the soundness of the semantics that every such set is a clique in the corresponding space.

There are two variants for the web of the exponentials : set based and multiset based. In the multiset based semantics, the web of $!X$ is the set of multisets which supports are cliques in X:

$$|!X| = \{\mu \in \mathcal{M}_{\mathrm{fin}}(|X|) \mid \mathrm{supp}(\mu) \in \mathrm{Cl}(X)\} \tag{4}$$

Two elements μ and ν of the web of $!X$ are coherent iff the support of $\mu + \nu$ is a clique of X.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \bot, 1 : \{(*,*)\}}(\mathit{ax.}) \quad \dfrac{}{\vdash \bot, 1 : \{\cancel{(*,*)}\}}(\mathit{ax.})}{\vdash \bot \,\&\, \bot, 1 : \{(\mathtt{v},*), \cancel{(\mathtt{f},*)}\}}(\mathit{avec})}{\vdash \bot \,\&\, \bot, 1, \bot : \{(\mathtt{v},*,*), \cancel{(\mathtt{f},*,*)}\}}(\mathit{bot}) \quad \dfrac{\dfrac{\dfrac{}{\vdash \bot, 1 : \{\cancel{(*,*)}\}}(\mathit{ax.}) \quad \dfrac{}{\vdash \bot, 1 : \{(*,*)\}}(\mathit{ax.})}{\vdash \bot \,\&\, \bot, 1 : \{\cancel{(\mathtt{v},*)}, (\mathtt{f},*)\}}(\mathit{avec})}{\vdash \bot \,\&\, \bot, 1, \bot : \{\cancel{(\mathtt{v},*,*)}, (\mathtt{f},*,*)\}}(\mathit{bot})}{\vdash \bot \,\&\, \bot, 1, \bot \,\&\, \bot : \{(\mathtt{v},*,\mathtt{v}), \cancel{(\mathtt{f},*,\mathtt{v})}, \cancel{(\mathtt{v},*,\mathtt{f})}, (\mathtt{f},*,\mathtt{f})\}}(\mathit{avec})}{\vdash ?(\bot \,\&\, \bot), 1, \bot \,\&\, \bot : \{([\mathtt{v}],*,\mathtt{v}), \cancel{([\mathtt{f}],*,\mathtt{v})}, \cancel{([\mathtt{v}],*,\mathtt{f})}, ([\mathtt{f}],*,\mathtt{f})\}}(\mathit{der.})}{\vdash ?(\bot \,\&\, \bot), 1, ?(\bot \,\&\, \bot) : \{([\mathtt{v}],*,[\mathtt{v}]), \cancel{([\mathtt{f}],*,[\mathtt{v}])}, \cancel{([\mathtt{v}],*,[\mathtt{f}])}, ([\mathtt{f}],*,[\mathtt{f}])\}}(\mathit{der.})}{\vdash ?(\bot \,\&\, \bot), 1 : \{([\mathtt{v},\mathtt{v}],*), \cancel{([\mathtt{f},\mathtt{v}],*)}, ([\mathtt{f},\mathtt{f}],*)\}}(\mathit{cont.})
$$

Figure 1: linear logic proof of Example 1

The interpretation of exponential rules is the same as in the relational semantics but restricted to the web of the exponentials. Two rules have their interpretation modified by this restriction. In the contraction rule, the support of $\mu_1 + \mu_2$ has to be an anti-clique of (the space interpreting) A. In the promotion rule : (i) the support of $\sum_{j \in J} \mu_i^j$ has to be an anti-clique of A_i, for each j; (ii) and the support of $[a_j \mid j \in J]$ has to be a clique of A. In fact, one can easily verify that (i) implies (ii) so the only condition to check is (i). This will also be the case in hypercoherences.

In the set based semantics, the web of $!X$ is the set of finite cliques of X. The interpretation of exponentials follows the last pattern but with sets and unions instead of mulitsets and sums.

Exemple 1. In the introduction we gave unformaly the relational interpretation and the uniform coherence space interpretation of a term. The proof in Figure 1 is a linear logic version of this term annotated by the relational interpretation of its sub-proofs. In this example we have denoted $(1, *)$ by `v` and $(2, *)$ by `f`. The point which is forgotten by the coherence space semantics of this proof and the points from which it comes from are printed with a line through text.

Exemple 2. Another example of the action of uniformity concerns restrictions on the number of times an argument will be copied. This is sligthly more subtle than just removing pieces of dead code.

Consider the following proof π_1 :

$$
\dfrac{\dfrac{}{\vdash 1}(\mathit{un})}{\vdash !1.}(\mathit{prom.})
$$

which intuitive meaning is that we make 1 available *ad libidum*. If we add a dereliction as last rule, the proof π_1 intuitively becomes a program taking as argument another program requiring an arbitrary number of copies of 1. For instance, the two points $([[*, *]])$ and $([[*]])$ in the interpretation of π_1 will corresponds to a required number of copies of, respectively, 2 and 1.

Now consider two copies of π_1 which we assemble into a unique proof by a combination of bottom and tensor introductions (we could also have used the *mix* rule if available). We contract the two copies of ?!1. The resulting proof π_2 is shown in Figure 2. In coherence spaces, the uniformity restriction forces the two copies of π_1 to receive each a program asking for 1 the *same number of times*. For instance, in coherence spaces: $([[*, *], [*, *]], (*, *))$ is in the interpretation of π_2 but $([[*], [*, *]], (*, *))$ is not; while, in the relational semantics, both of these points are in the interpretation.

2.5 Hypercoherences

An hypercoherence is a reflexive $\mathcal{P}^*_{\mathrm{fin}}$-coherence space. The interpretation of linear logic in hypercoherences [Ehr93] follows exactly the same pattern as for coherence spaces. We just stress a few points.

In a tensor one has $x = \{(a_1, b_1), \ldots, (a_n, b_n)\} \in \frown_{X \times Y}$ iff $\pi_1 x = \{a_1, \ldots, a_n\} \in \frown_X$ and $\pi_2 x = \{b_1, \ldots, b_n\} \in \frown_Y$. In a *with*, the dual of a *plus*, $\asymp_{A\&B} = \asymp_A \uplus \asymp_B$, hence for every $x \in \mathcal{P}^*_{\mathrm{fin}}(|A| + |B|)$, if x intersects both $|A|$ and $|B|$ then $x \in \frown_{A\&B}$ (and conversely).

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\,(un)}{\vdash !1}\,(prom.)}{\vdash ?!1}\,(der.)}{\vdash ?!1, \bot}\,(bot) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1}\,(un)}{\vdash !1}\,(prom.)}{\vdash ?!1}\,(der.)}{\vdash ?!1, \bot}\,(bot)}{\vdash ?!1, ?!1, \bot \otimes \bot}\,(tens.)}{\vdash ?!1, \bot \otimes \bot}\,(cont.)
$$

Figure 2: Proof of Example 2

The two variants for the exponentials, set based and multiset based, are also presents.

The coherence in $!X$ is defined using a notion of section. If $U = \{x_i \mid i \in I\}$ is a finite set of finite sets or of multisets we say that s is a *section* of U and write $s \lhd U$ when for each $i \in I$ there exists $a_i \in s$ such that $a_i \in x_i$ and $s \subseteq \cup_{i \in I} x_i$. A non empty finite subset U of $|!X|$ is coherent in $!X$ iff each section of U is coherent in X.

Of course, since the notion of coherence differs between coherence spaces and hypercoherence, the notion of cliques and so, because of uniformity, the interpretation of proofs also differ.

Note that the coherence in hypercoherence may have *holes* : in general, one can have $x \in \frown_X$ and $y \subset x$ without having $y \in \frown_X$.

Propriété 2: *In hypercoherence and coherence space semantics, the intersection of a clique and of an anti-clique contains at most one point (determinism). But, moreover, in these two semantics, if $(a, c) \in f \mathbin{;} g$ for $f : A \to B$ and $g : B \to C$ then there is only one b such that $(a, b) \in f$ and $(b, c) \in g$.*

Let us recall that one cannot equip the relational semantics with a set based exponential ($|!X| = \mathcal{P}_{\mathrm{fin}}(|X|)$) similar to the one of coherence spaces and hypercoherences. This will not give a sound interpretation. Consider for instance the diagram setting the naturality of dereliction

$$
\begin{tikzcd}
!X \arrow[r, "\mathrm{der}_X"] \arrow[d, "!f"'] & X \arrow[d, "f"] \\
!Y \arrow[r, "\mathrm{der}_Y"'] & Y
\end{tikzcd}
$$

In the particular case where $f = \{(a, b), (a', b)\}$, one will have $(\{a, a'\}, \{b\}) \in !f$ so $(\{a, a'\}, b) \in !f \mathbin{;} \mathrm{der}_Y$ but $(\{a, a'\}, b) \notin \mathrm{der}_X \mathbin{;} f$. Hence the diagram won't commute. Saying it in a category free manner, with such a set based exponentials, the elimination of a cut between a promotion and a dereliction won't, in general, leave the interpretation invariant.

3 Bipartite uniform and non uniform relational semantics

In this section, we introduce a simple semantics of linear logic, based on the relational semantics and id-coherence spaces, and which admits both a non uniform version and a uniform version. We use these two semantics to demonstrate that uniform semantics can lose a lot of information about terms (proofs) they interpret.

Définition 1. A *bipartite space* is just a id-coherence space $(|X|, \frown_X, \asymp_X)$ where N_X is empty. So $\frown_X$, $\asymp_X$ is a bipartition of $|X|$ and every bipartition of $|X|$ defines a bipartite space.

We further call positive web, denoted $|X|^+$, the coherence of X and negative web, denoted $|X|^-$, the incoherence of X. The elements of $|X|^+$ (resp. $|X|^-$) are the positive (resp. negative) points of X.

Following the general definition (Def. 3) in our particular case, a clique is just a set of positive points of the web.

The category **Bip** has bipartite spaces as objets and for morphisms between bipartite spaces X and Y, the relations which are cliques in the bipartite space $X \multimap Y$ defined below. Identities and composition are those of **Rel**. We now describe the non uniform bipartite semantics of linear logic.

On formulæ the logical connectives are interpreted as follows:

- linear negation is given, as in the general P-cohérence space case, by the exchange of coherence and incoherence, so $X^{\perp} = (|X|, |X|^{-}, |X|^{+})$;
- both additives constants 0 and $\top$ are equal to $(\emptyset, \emptyset, \emptyset)$;
- the bipartite space 1 is equal to $(\{*\}, \{*\}, \emptyset)$ and, so the bipartite space $\perp$ is equal to $(\{*\}, \emptyset, \{*\})$;
- the *with* is given by $X \& Y = (|X| + |Y|, |X|^{+} + |Y|^{+}, |X|^{-} + |Y|^{-})$ and the *plus* is given by $X \oplus Y = (X^{\perp} \& Y^{\perp})^{\perp}$ which is the same bipartite space as $X \& Y$;
- the tensor of X and Y, $X \otimes Y$, is the bipartite space $|X| \times |Y|$ whose positive web $|X \times Y|^{+}$ is equal to $|X|^{+} \times |Y|^{+}$. It follows that $X ⅋ Y = (X^{\perp} \otimes Y^{\perp})^{\perp}$ is such that (a, b) is positive in $X ⅋ Y$ iff a or b is positive and that $X \multimap Y = X^{\perp} ⅋ Y$ is such that $|X \multimap Y| = |X| \times |Y|$ and $(a, b) \in |X \multimap Y|^{+}$ iff $a \in |X|^{+}$ implies $b \in |Y|^{+}$.
- The *of course* of X, $!X$ is the bipartite space of web $\mathcal{M}_{\mathrm{fin}}|X|$ and of positive web $|!X|^{+} = \mathcal{M}_{\mathrm{fin}}|X|^{+}$. Thus a multiset μ, element of $|!X|$, is negative iff μ contains at least one negative point of X. It follows that $?X = (!X^{\perp})^{\perp}$ is such that an element μ of $|?X|$ is positive iff it contains at least one positive point of X.

Remarque. In contrast to the relational semantics, the bipartite semantics distinguishes A and $A^{\perp}$. In particular the multiplicative constants are distinct (this is not the case in hypercoherences).

As usual a context $A_1, \ldots, A_n$ is interpreted by the same space as the formula $A_1 ⅋ \ldots ⅋ A_n$. Interpretations of proofs are defined as in the relational semantics. One easily verifies that a proof is interpreted by a set of positive points (a clique). The categorical structure of the bipartite semantics is derived from the one of the relational semantics. Morphisms involved in natural transformations of the semantics and morphisms obtained by functorial constructions, seen as sets, are defined the same as in the relational semantics and it is straightforward to verify that they actually contain only positive points, so they are cliques.

3.1 Uniform bipartite semantics

We introduce a uniform variant of the bipartite semantics as follows. The uniform interpretation of exponentials is given by a comonad $(\underset{u}{!}, \mathrm{der}_u, \mathrm{dig}_u)$ described below. The others categorical constructions are the same as in the non uniform bipartite semantics.

Setting $\mathrm{Pos}(X) = (|X|^{+}, |X|^{+}, \emptyset)$ for each bipartite space X and $\mathrm{Pos}(f) = f \cap (|X|^{+} \times |Y|^{+})$ for each $f \in \mathbf{Bip}(X, Y)$ trivially makes Pos a functor.

Since f is a clique in $X \multimap Y$, if $(a, b) \in f$ and $a \in |X|^{+}$ then $b \in |X|^{+}$. Hence $\mathrm{Pos}(f)$ is equal as a set with $\mathrm{Pos}(\mathrm{id}_X) \mathbin{⨾} f$, where $\mathrm{Pos}(\mathrm{id}_X)$ shall be seen by set inclusion as a morphism from $\mathrm{Pos}(X)$ to X. This can be used to verify that Pos commutes with the composition: if $f \in \mathbf{Bip}(X, Y)$ and $g \in \mathbf{Bip}(Y, Z)$ then we have the following set equalities

$$\begin{aligned}
\mathrm{Pos}(f) \mathbin{⨾} \mathrm{Pos}(g) &= \mathrm{Pos}(f) \mathbin{⨾} \mathrm{id}_{\mathrm{Pos}(\mathrm{id}_Y)} \mathbin{⨾} g \\
&= \mathrm{Pos}(f) \mathbin{⨾} g \\
&= \mathrm{Pos}(\mathrm{id}_X) \mathbin{⨾} f \mathbin{⨾} g \\
&= \mathrm{Pos}(f \mathbin{⨾} g).
\end{aligned}$$

The functor $\underset{u}{!}$ is defined by setting $\underset{u}{!} = \mathrm{Pos}\,!$ (where $!$ is the functor "of course" of the non uniform bipartite semantics). Remark that $\mathrm{Pos}\,! = \,!\,\mathrm{Pos}$. The natural transformations $\mathrm{der}_u : \underset{u}{!} \dot{\to} \mathrm{id}$ and $\mathrm{dig}_u : \underset{u}{!} \dot{\to} \underset{u}{!}\,\underset{u}{!}$ are defined by setting $\mathrm{der}_{u,X} = \{[a] \mid a \in |X|^+\}$ (which is equal as a set to $\mathrm{der}_{\mathrm{Pos}(X)}$ and $\mathrm{Pos}\,\mathrm{der}_X$) and $\mathrm{dig}_u = \mathrm{Pos}(\mathrm{dig})$ (one has $\mathrm{Pos}\,!! = \underset{u}{!}\,\underset{u}{!}$). We have to verify that der_u is actually a natural transformation (for dig_u this follows from the definition). Let $f \in \mathbf{Bip}(X,Y)$. Then

$$\mathrm{der}_{u,X} \mathbin{;} f = \mathrm{der}_{u,x} \mathbin{;} \{(a,b) \mid a \in |X|^+\} = \mathrm{der}_{u,X} \mathbin{;} \mathrm{Pos}(\mathrm{id}_X) \mathbin{;} f$$

which is equal as a set with

$$\mathrm{Pos}(\mathrm{der}_X \mathbin{;} f) = \mathrm{Pos}(!f \mathbin{;} \mathrm{der}_Y) = \underset{u}{!}\, f \mathbin{;} \mathrm{der}_{u,Y}\,.$$

And this concludes. The followings commutative diagrams are the image by Pos of the corresponding commutative diagrams for $(!, \mathrm{der}, \mathrm{dig})$ in $\mathbf{Bip}$.

$$\begin{tikzcd}
\underset{u}{!}\,E \arrow[r, "\mathrm{dig}_E"] \arrow[dr, "\underset{u}{!}\,I_E"'] & \underset{u}{!}\,\underset{u}{!}\,E \arrow[d, "\underset{u}{!}\,\mathrm{der}_{u,E}"] \\
& \underset{u}{!}\,E
\end{tikzcd}
\qquad
\begin{tikzcd}
\underset{u}{!}\,E \arrow[r, "\mathrm{dig}_{u,E}"] \arrow[dr, "\underset{u}{!}\,I_E"'] & \underset{u}{!}\,\underset{u}{!}\,E \arrow[d, "\mathrm{der}_{u,\underset{u}{!}\,E}"] \\
& \underset{u}{!}\,E
\end{tikzcd}
\qquad
\begin{tikzcd}
\underset{u}{!}\,E \arrow[r, "\mathrm{dig}_{u,E}"] \arrow[d, "\mathrm{dig}_{u,E}"'] & \underset{u}{!}\,\underset{u}{!}\,E \arrow[d, "\underset{u}{!}\,\mathrm{dig}_{u,E}"] \\
\underset{u}{!}\,\underset{u}{!}\,E \arrow[r, "\mathrm{dig}_{u,\underset{u}{!}\,E}"'] & \underset{u}{!}\,\underset{u}{!}\,\underset{u}{!}\,E
\end{tikzcd}$$

Hence $(\underset{u}{!}, \mathrm{der}_u, \mathrm{dig}_u)$ is truly a comonad.

To achieve the verification that this setting form a new Seely categorical semantics of linear logic one has to verify that the adjunction induced by the comonad $(\underset{u}{!}, \mathrm{der}_u, \mathrm{dig}_u)$ is monoidal. We won't check this in detail but it can be easily derived from the fact that in $\mathbf{Rel}$ the comonad $(!, \mathrm{der}, \mathrm{dig})$ induces a monoidal adjunction. Just remark that the isomorphisms $\underset{u}{!}(X \mathbin{\&} Y) \cong \underset{u}{!}(X) \otimes \underset{u}{!}(Y)$ and $\underset{u}{!}\,\top \cong 1$ hold and are natural (since $\mathrm{Pos}(X') \otimes \mathrm{Pos}(Y') \cong \mathrm{Pos}(X' \otimes Y')$, $\mathrm{Pos}(f) \otimes \mathrm{Pos}(g)$ is the same set as $\mathrm{Pos}(f \otimes g)$ and $\mathrm{Pos}(1) = 1$).

The interpretation of exponential rules in the uniform bipartite semantics is obtained by a set restriction to the uniform web(s), as follows.

The interpretations of the two rules contraction and weakening, are, in fact, unchanged:

$$\frac{\vdash \Gamma, ?A, ?A : f}{\vdash \Gamma, ?A : \{(\gamma, \mu_1 + \mu_2) \mid (\gamma, \mu_1, \mu_2) \in f\}}\ (\mathit{cont.}) \qquad \frac{\vdash \Gamma : f}{\vdash \Gamma, ?A : \{(\gamma, []) \mid (\gamma) \in f\}}\ (\mathit{aff.})$$

because in the sets produced by these two rules there are already only negative points in $?A$. In the contraction rule, μ_1 and μ_2 contains only negative points of the interpretation of A, so does $\mu_1 + \mu_2$. And for the weakening rule we have that $[]$ is negative in $?A$.

Thus, in contrast to the coherence space situation (for instance), the construction associated with the contraction is the same as in the relational semantics.

The interpretation of the dereliction rule is:

$$\frac{\vdash \Gamma, A : f}{\vdash \Gamma, ?A : \{(\gamma, [a]) \mid (\gamma, a) \in f, a \in |A|^-\}}\ (\mathit{der.})$$

(where $|A|^-$ stands for $|X|^-$ with X interpreting A). Remark that this construction forgets some points relative to the non uniform interpretation (contrarily to what happens in coherence spaces).

The interpretation of the promotion rule is:

$$\frac{\vdash ?A_1, \ldots, ?A_n, A : f}{\vdash ?A_1, \ldots, ?A_n, !A : f^{\dagger_u}}\ (\mathit{prom.})$$

where $f^{\dagger_u}$ is equal to the restriction to the uniform web of the $f^\dagger$ of the relational semantics. But again (contrarily to what happens in coherence spaces) there is no need to restrict and so:

$$f^{\dagger_u} = f^\dagger = \{(\sum_{j\in J}\mu_1^j, \ldots, \sum_{j\in J}\mu_n^j, [a_j \mid j \in J]) \mid [(\mu_1^j, \ldots, \mu_n^j, a_j) \mid j \in J] \in \mathcal{M}_{\mathrm{fin}}(f)\}.$$

This is because of two reasons. First, the sum of multisets of negative points is a multiset of negative points, so $\sum_{j\in J}\mu_i^j$ is in $|?A_i|^-$, for each i. Second, since f is a clique, each element $(\mu_1, \ldots, \mu_n, a)$ of f is positive. Since each μ_i is negative in $?A_i$ it follows that a is positive in $!A$ and so $[a_j \mid j \in J] \in |!A|^+$.

Remarque. In the uniform bipartite semantics, each proof π of a sequent $\vdash ?A$ is interpreted by the empty set.

To state this remark simply observe that the space interpeting $?A$ contains only negative points and that a clique is a set of positive points.

Exemple 1. The interpretation of the proof

$$\dfrac{\dfrac{}{\vdash \perp, 1}\,(ax.)}{\vdash ?1, \perp}\,(der.)$$

is the empty set. But the interpretation of the proof of Figure 1 is the same as in the relational semantics.

Curiously enough the uniform bipartite semantics maps a lot of proofs to the empty set. (But many other proofs are mapped on non trivial subsets of their relational interpretations).

3.2 Extensional collapses

The basis types **bool** and **nat** are interpreted by the respective bipartite spaces:

$$\mathbf{bool} = (\{\mathtt{v}, \mathtt{f}\}, \{\mathtt{v}, \mathtt{f}\}, \emptyset) \text{ et}$$
$$\mathbf{nat} = (\mathbb{N}, \mathbb{N}, \emptyset).$$

Simple types are interpreted by the same bipartite spaces in the uniform bipartite semantics and in the non uniform bipartite semantics. Moreover, the bipartite spaces interpreting simple types are purely positive (every point of the web is positive) so every subset of the web is a clique. Hence the uniform bipartite semantics, the non uniform bipartite semantics and the relational semantics have the same extensional collapse. We don't know any direct expression of this collapse.

4 Non uniform K-cohérent semantics

4.1 K-coherence spaces

From now on, we shall assume that a subset K of $\mathbb{N} \setminus \{0, 1\}$ is given, and we call the corresponding $\mathcal{M}_K$-coherence space a K-cohérent space.

4.2 Interpreting MALL... nothing new

The interpretation of the multiplicative additive fragment of linear logic (MALL) follows a standard pattern.

Both additive constants are the empty K-cohérent space:

$$0 = \top = (\emptyset, \emptyset, \emptyset).$$

Both multiplicative constants are the reflexive one point K-cohérent space

$$1 = \perp = (\{*\}, \mathcal{M}_K(\{*\}), \mathcal{M}_K(\{*\})).$$

Let X_1 and X_2 be two K-cohérent spaces.

- The K-cohérent space $X_1 \oplus X_2$ is defined by setting

$$\begin{aligned}|X_1 \oplus X_2| &= |X_1| + |X_2|,\\ \mathsf{N}_{X_1 \oplus X_2} &= \mathsf{N}_{X_1} \uplus \mathsf{N}_{X_2} \text{ et}\\ \frown_{X_1 \oplus X_2} &= \frown_{X_1} \uplus \frown_{X_2}.\end{aligned}$$

  Of course $X_1 \& X_2 = ({X_1}^\perp \oplus {X_2}^\perp)^\perp$.

- The space $X_1 \otimes X_2$ is defined as follows. We set $|X_1 \otimes X_2| = |X_1| \times |X_2|$. For $i = 1, 2$, let π_i be the canonical projections:

$$\begin{aligned}\pi_i : \mathcal{M}_K(|X_1 \otimes X_2|) &\to \mathcal{M}_K(|X_i|)\\ [(a_j^1, a_j^2) \mid j \in J] &\mapsto [a_j^i \mid j \in J].\end{aligned}$$

  Then for each $s \in \mathcal{M}_K(|X_1 \otimes X_2|)$ we set

$$\begin{aligned}s \in \mathsf{N}_{X_1 \otimes X_2} &\text{ ssi } \pi_1(s) \in \mathsf{N}_{X_1} \text{ et } \pi_2(s) \in \mathsf{N}_{X_2}\\ s \in \smile_{X_1 \otimes X_2} &\text{ ssi } \pi_1(s) \in \smile_{X_1} n \text{ ou } \pi_2(s) \in \smile_{X_2}\end{aligned}$$

  which suffices to determine $\mathsf{N}_{X_1 \otimes X_2}$, $\smile_{X_1 \otimes X_2}$ and $\frown_{X_1 \otimes X_2}$. We also set $X_1 \parr X_2 = ({X_1}^\perp \otimes {X_2}^\perp)^\perp$.

The *linear map* construction $\multimap$ between K-cohérent spaces is defined by setting $X \multimap Y = X^\perp \parr Y$. A *linear morphism* from X to Y, two K-cohérent spaces, is a clique of $X \multimap Y$. Remark that

$$s \in \mathrel{\overset{\frown}{\smile}}_{X \multimap Y} \text{ ssi } \begin{cases}\pi_1(s) \in \mathrel{\overset{\frown}{\smile}}_X \implies \pi_2(s) \in \mathrel{\overset{\frown}{\smile}}_Y\\ \pi_1(s) \in \frown_X \implies \pi_2(s) \in \frown_Y\end{cases} \tag{5}$$

or equivalently,

$$s \in \mathrel{\overset{\frown}{\smile}}_{X \multimap Y} \text{ ssi } \begin{cases}\pi_2(s) \in \smile_Y \implies \pi_1(s) \in \smile_X\\ (\pi_1(s) \in \mathrel{\overset{\frown}{\smile}}_X \text{ et } \pi_2(s) \in \mathsf{N}_Y) \implies \pi_1(s) \in \mathsf{N}_X.\end{cases} \tag{6}$$

or equivalently,

$$s \in \mathrel{\overset{\frown}{\smile}}_{X \multimap Y} \text{ ssi } \begin{cases}\mathrel{\overset{\frown}{\smile}}_X \pi_1(s) \implies \mathrel{\overset{\frown}{\smile}}_Y \pi_2(s)\\ \asymp_Y \pi_2(s) \implies \asymp_X \pi_1(s)\end{cases} \tag{7}$$

We denote by $\mathbf{NCOH}_K$ the category whose objects are the K-cohérent spaces, whose morphisms are the linear morphisms and where compositions and identities are defined as in **Rel** (one easily verifies that the composition of a clique of $X \multimap Y$ and a clique of $Y \multimap Z$ is a clique). For every $K' \subseteq K$, the corresponding categories come naturally with forgetful functors $U_{K,K'} : \mathbf{NCOH}_K \to \mathbf{NCOH}_{K'}$ which act as the identity on morphisms.

The *boolean type*, denoted by **bool** and represented by the formula $1 \oplus 1$ will be interpreted, in $\mathbf{NCOH}_{\mathbb{N}\setminus\{0,1\}}$, by the uniform $\mathbb{N} \setminus \{0, 1\}$-cohérent space whose web is $\{\mathtt{v}, \mathtt{f}\}$ and whose coherence is $\mathcal{M}_{\mathbb{N}\setminus\{0,1\}}(\{\mathtt{v}\}) \cup \mathcal{M}_{\mathbb{N}\setminus\{0,1\}}(\{\mathtt{f}\})$.

Proposition 1 (semantics of MALL). *For each $K \subseteq \mathbb{N}\setminus\{0, 1\}$, the category $\mathbf{NCOH}_K$ is a semantics of MALL. And for each $K' \subseteq K$ (in particular for $K' = \emptyset$) the functor $U_{K,K'}$ is logical w.r.t. the $\mathbf{NCOH}_K$ and $\mathbf{NCOH}_{K'}$ MALL semantics (logicial means that, commutes to the interpretations of sequents and proofs).*

The fact that $\mathbf{NCOH}_K$ is a semantics of MALL means that $\mathbf{NCOH}_K$ is a symmetric monoidal closed category (with $\otimes$ as tensor product and $\multimap$ as function space constructor) which is $*$-autonomous ($\perp$ being the dualizing object), and furthermore, has all finite products and coproducts (see [AC98] for precise definitions). The proof, sketched below, is a straightforward verification.

Proof. The only thing to verify is that all the morphisms of **Rel** which make **Rel** a $*$-autonomous category, are cliques, that is linear morphism of the category $\mathbf{NCOH}_K$. Provided Y_1 and Y_2 are of disjoint web, $Y_1 \& Y_2$ equipped with the two projections

$$p_i = \{(a,a) \mid a \in |Y_i|\} : (Y_1 \& Y_2) \to Y_i$$

for $i = 1,2$ is the Cartesian product of Y_1 and Y_2 and if $f_1 \in \mathbf{NCOH}_K(X, Y_1)$ and $f_2 \in \mathbf{NCOH}_K(X, Y_2)$ then the *pairing* of f_1 and f_2 is just $(f_1 \cup f_2) \in \mathbf{NCOH}_K(X, Y_1 \& Y_2)$. In fact one easily verifies that p_1, p_2 and $f_1 \cup f_2$ are cliques. For each $f \in \mathbf{NCOH}_K(X, Y)$, and $f' \in \mathbf{NCOH}_K(X', Y')$,

$$f \otimes f' = \{((a,a'),(b,b')) \mid (a,b) \in f, (a',b') \in f'\}$$

is obviously a clique of $(X \otimes X') \multimap (Y \otimes Y')$, thus the construction $\otimes$ on K-cohérent spaces is functorial. For each $X, Y, Z \in \mathbf{NCOH}_K$, the isomorphisms $\mathrm{unit}_X : X \otimes 1 \cong X$, $\mathrm{ass}_{X,Y,Z} : (X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$ and $\mathrm{com}_X : X \otimes X \to X \otimes X$ of **Rel** given by

$$\begin{aligned}
\mathrm{unit}_X &= \{((a,*),a) \mid a \in X\}, \\
\mathrm{ass}_{X,Y,Z} &= \{(((a,b),c),(a,(b,c))) \mid a \in X, b \in Y, c \in Z\} \text{ and} \\
\mathrm{com}_X &= \{((a,b),(b,a)) \mid a,b \in X\},
\end{aligned}$$

are easily verified to be also isomorphisms in $\mathbf{NCOH}_K$. So together with the functor $\otimes$ it gives a symmetric monoidal structure on $\mathbf{NCOH}_K$ which turns to be closed for the built-in object of morphisms $X \multimap Y$. Finally the dualising object $\perp$ is clearly such that $X \multimap \perp \cong X^\perp$ thus $X^{\perp\perp}$ is obviously isomorphic to X. $\square$

Remarque (foliation). The coherence relation is *foliated* with respects to the interpretation of MALL *i.e.* for each formula A of MALL the coherence relations on multisets of cardinality n in the interpretation of A is totally determined by the coherence relations on multisets of cardinality n in the interpretation of the sub-formulae of A. In fact, this is exactly by constructing independently each coherence relation of level k for $k \in K$ in the Bucciarelli-Ehrhard machinery that the K-cohérent spaces semantics has been obtained, so this remark also holds for the linear logic semantics with the exponential provided by this machinery. Anticipating a bit, it will also hold for the new exponential construction we present (and the forgetful functors $U_{K,K'}$ will still be logical in LL).

4.3 Exponentials

Using the constructions presented by Bucciarelli and Ehrhard in [BE01], one can define exponentials for K-cohérent spaces.

This gives a semantics which accepts a variant of the well known Berry's example of a stable and non sequential function from **bool** $\times$ **bool** $\times$ **bool** to **bool**. Of course both the standard (set based) hypercoherence semantics and the multiset based hypercoherence semantics we use here reject such first order non-sequential functions.

The corresponding "of course" operation is denoted by $\underset{i}{!}$ and defined as follows. We set $|\underset{i}{!}X| = \mathcal{M}_{\mathrm{fin}}(|X|)$. A multiset $[x_i \mid 1 \leq i \leq k] \in \mathcal{M}_K(|\underset{i}{!}X|)$ (so that $k \in K$) is strictly incoherent in $\underset{i}{!}X$ iff there exists a multiset $[a_j \mid 1 \leq j \leq k] \in \mathcal{M}_K(|X|)$ which is strictly incoherent in X and satisfies

$$[a_j \mid 1 \leq j \leq k] \leq \sum_{i=1}^{k} x_i.$$

If such a multiset $[a_j \mid 1 \leq j \leq k]$ does not exist, $[x_i \mid 1 \leq i \leq k]$ is coherent and then, it is strictly coherent exactly when $\sum_{i=1}^{k} x_i$ is *star-shaped* that is when there exists $a \in |\sum_{i=1}^{k} x_i|$ such that

$$\forall (a_j)_{1 \leq j \leq k} \in |X|^k, ([a_j \mid 1 \leq j \leq k] \leq \sum_{i \in I} x_i \text{ et } a_k = a) \implies [a_j \mid 1 \leq j \leq k] \in \frown_X .$$

For instance $\underset{i}{!}\mathbf{bool}$ is given by:

$$
\begin{aligned}
[x_i \mid i \in I] \in \smile_{\underset{i}{!}\mathbf{bool}} \text{ ssi } & \sum_{i\in I} x_i = p[\mathtt{v}] + q[\mathtt{f}] \text{ with } p, q > 0 \text{ et } p + q \geq \sharp I \\
[x_i \mid i \in I] \in \frown_{\underset{i}{!}\mathbf{bool}} \text{ ssi } & \sum_{i\in I} x_i = p[\mathtt{v}] + q[\mathtt{f}] \text{ with } 1 \leq p + q < \sharp I \\
\text{hence } [x_i \mid i \in I] \in \mathsf{N}_{\underset{i}{!}\mathbf{bool}} \text{ ssi } & \sum_{i\in I} x_i = [], k[\mathtt{v}] \text{ ou } k[\mathtt{f}] \text{ with } k \geq \sharp I
\end{aligned}
$$

Consider the following subset of $|(\underset{i}{!}\mathbf{bool} \otimes \underset{i}{!}\mathbf{bool} \otimes \underset{i}{!}\mathbf{bool}) \multimap \mathbf{bool}|$:

$$
f = \{ \begin{array}{llll}
(([], & [\mathtt{v},\mathtt{v}], & [\mathtt{f},\mathtt{f}]), & \mathtt{v}), \\
(([\mathtt{f},\mathtt{f}], & [], & [\mathtt{v},\mathtt{v}]), & \mathtt{v}), \\
(([\mathtt{v},\mathtt{v}], & [\mathtt{f},\mathtt{f}], & []), & \mathtt{v})
\end{array} \}.
$$

It is a variant of the well known Berry's example of a stable and non sequential function from $\mathbf{bool}\times\mathbf{bool}\times\mathbf{bool}$ to $\mathbf{bool}$. This function is not a morphism in the multiset based hypercoherence semantics but the $\mathbb{N}\backslash\{0,1\}$-cohérent semantics with the $\underset{i}{!}$ exponential accepts it [2]. Indeed, for each multiset $s \in \mathcal{M}_K(f)$, on has $\pi_2(s) \in \mathcal{M}_K(\{\mathtt{v}\})$ thus $\pi_2(s) \in \mathsf{N}_{\mathbf{bool}}$. With respect to Equation 5, the only way for s to be strictly incoherent in $(\underset{i}{!}\mathbf{bool}\otimes\underset{i}{!}\mathbf{bool}\otimes\underset{i}{!}\mathbf{bool}) \multimap \mathbf{bool}$ is to have $\pi_1(s) \in \frown$ in $\underset{i}{!}\mathbf{bool} \otimes \underset{i}{!}\mathbf{bool} \otimes \underset{i}{!}\mathbf{bool}$. But if m_1, m_2 and m_3 are the respective numbers of occurrences of points of f in s then

- if only one of the m_i is non-zero then each of the projection of $\pi_1(s)$ on the three arguments is neutral so $\pi_1(s)$ is neutral;
- if exactly two of the m_i are non-zero (say m_1 and m_2) then the empty multiset does not occurs in one of the three projections of $\pi_1(s)$ (here the third) thus the sum of the multisets occurring in this projection contains enough $\mathtt{v}$ and $\mathtt{f}$ (here $2m_1\mathtt{f}$ and $2m_2\mathtt{v}$) comparatively to its cardinality (here $m_1 + m_2$) as to make it strictly incoherent in $\mathbf{bool}$. Hence $\pi_1(s)$ is surely strictly incoherent;
- finally, if none of the m_i is zero then $\pi_1(s)$ is coherent iff each of its three projections on $\underset{i}{!}\mathbf{bool}$ is coherent. That is iff

  $$
  \begin{aligned}
  m_1 + m_2 + m_3 &> 2m_2 + 2m_3 \text{ (coherence on the first argument)} \\
  m_1 + m_2 + m_3 &> 2m_1 + 2m_3 \text{ (coherence on the second argument)} \\
  m_1 + m_2 + m_3 &> 2m_1 + 2m_2 \text{ (coherence on the third argument)}
  \end{aligned}
  $$

  but we will then have that $3(m_1 + m_2 + m_3) > 4(m_1 + m_2 + m_3)$ which is impossible.

Thus f is definitely a clique of $(\underset{i}{!}\mathbf{bool} \otimes \underset{i}{!}\mathbf{bool} \otimes \underset{i}{!}\mathbf{bool}) \multimap \mathbf{bool}$.

The really surprising fact is that one can easily try to correct this by choosing another definition for the coherence relations of the exponential construction and obtain in that way a new semantics of linear logic. Among these variants for the exponentials there is a *most general one* in a sense which will be made precise in Theorem 1 and Corollary 1. Indeed the definition is guided by the need of Theorem 1.

First of all we adapt the notion of section of hypercoherences to the K-cohérent spaces setting.

Définition 1. If $\mu = [x_i \mid i \in I]$ is a multiset of finite sets or of multisets and if s is another multiset we say that s is a *section* of μ and we write $s \lhd \mu$ when there exists a family $(a_i)_{i\in I}$ such that $\forall i \in I, a_i \in x_i$ and $s = [a_i \mid i \in I]$ (in particular s and μ have the same cardinality).

[2] Remark that the same function with $[\mathtt{v}]$ instead of $[\mathtt{v},\mathtt{v}]$ and $[\mathtt{f}]$ instead of $[\mathtt{f},\mathtt{f}]$, is rejected by this semantics.

The notion of section between *sets* we used until now for hypercoherences can be rephrased by saying that a set s is a section of a set x iff there exists two multisets μ and ν such that $\nu \lhd \mu$, $\mathrm{supp}(\mu) = x$ and $\mathrm{supp}(\nu) = s$. We use the same name (section) but a different notation for the two notions: $\triangleleft$ between sets, $\lhd$ between multisets.

Définition 2. For each K-cohérent space X we define the K-cohérent space $!X$ as follows. Its web is $|!X| = \mathcal{M}_{\mathrm{fin}}(|X|)$ and for each element $[x_i \mid i \in I]$ of $\mathcal{M}_K(|!X|)$ we set:

$$[x_i \mid i \in I] \in \smile_{!X} \text{ ssi } \exists (a_i)_{i\in I}, [a_i \mid i \in I] \in \smile_X \text{ et } \forall i \in I, a_i \in x_i \tag{8}$$

and

$$[x_i \mid i \in I] \in \mathsf{N}_{!X} \text{ ssi } \begin{cases} [x_i \mid i \in I] \notin \smile_{!X} \text{ et} \\ \exists (a_i^j)_{i\in I}^{j\in J}, \begin{cases} \forall i \in I, [a_i^j \mid j \in J] = x_i \text{ et} \\ \forall j \in J, [a_i^j \mid i \in I] \in \mathsf{N}_X \end{cases} \end{cases} \tag{9}$$

We also define $?X$ by setting $?X = (!X^\perp)^\perp$.

When $K = \emptyset$, the exponential construction on objects is the standard exponential of **Rel**.

Exemple 1. The coherence in $!\mathbf{bool}$ is as follows. For $\mu \in \mathcal{M}_K(|\mathbf{bool}|)$

$$\mu \in \mathrel{\overset{\frown}{\smile}}_{!\mathbf{bool}} \text{ ssi } \begin{cases} \mathrm{supp}(\mu) \subset \mathcal{M}_{\mathrm{fin}}(\{\mathtt{v}\}) \text{ ou} \\ \mathrm{supp}(\mu) \subset \mathcal{M}_{\mathrm{fin}}(\{\mathtt{f}\}) \text{ ou} \\ [] \in \mu \end{cases}$$

$$\mu \in \mathsf{N}_{!\mathbf{bool}} \text{ ssi } \mathrm{supp}(\mu) = \{k[\mathtt{v}]\} \text{ ou } \{k[\mathtt{f}]\}, k \in \mathbb{N}.$$

Thanks to this exponential, our variant of the Berry's example is successfully rejected : if $3 \in K$, then f (as previously defined) is not a clique. Take $[a, b, c] \in \mathcal{M}_K(f)$ where a, b and c are the three points of f, then each of the three projections of $[a, b, c]$ on $!\mathbf{bool}$ is strictly coherent so is the projection on $!\mathbf{bool} \otimes !\mathbf{bool} \otimes !\mathbf{bool}$ but the projection on $\mathbf{bool}$ is neutral thus $[a, b, c] \in \smile_X$.

Exemple 2. Consider the K-cohérent space G with web $|G| = \{a, b, c\}$ and such that if $u \in \mathcal{M}_{\mathrm{fin}}(|G|)$ then: $u \in \mathsf{N}_G$ iff $\mathrm{supp}(u)$ is a singleton, $u \in \frown_G$ iff $\sharp\mathrm{supp}(u) = 2$ and $u \in \smile_G$ iff $\mathrm{supp}(u) = \{a, b, c\}$. The space G is in fact the sub-space of $\mathbf{bool}^3 \to \mathbf{bool}$ of web (the variant of) the Berry's example f above.

Suppose $2 \in K$. All the sections of $[[a], [b, c]]$ are coherent in G moreover $[a]$ and $[b, c]$ have not the same cardinality. So $[[a], [b, c]] \in \frown_{!G}$. Now suppose $3 \in K$. Then $[[a], [b, c], [b, c]]$ admits the strictly incoherent section $[a, b, c]$ but $[[a], [a], [b, c]]$ not and so $[[a], [b, c], [b, c]] \in \smile_{!G}$ but $[[a], [a], [b, c]] \in \frown_{!G}$. So the coherence relations of $!G$ depends on multiplicities.

For each $k \in K$ such that $k \geq 3$, each $m \in \mathcal{M}_{\{k\}}(|G|)$ such that $\mathrm{supp}(m) = \{[a, b], [a, c]\}$ is strictly incoherent in $!G$ but if $2 \in K$, $[[a, b], [a, c]] \in \frown_{!G}$ (all the sections of $[[a, b], [a, c]]$ are coherent in G and b is not neutral with any element of $[a, c]$).

Finally $[[a, b, c], [a, b, c], [a, b, c]]$ is an example of a non neutral (strictly incoherent, here) multiset in $!G$ of support a singleton.

Proposition 2 (semantics of LL). *Any category* $\mathbf{NCOH}_K$ *with the exponentials of Definition 2 is a semantics of linear logic (see [AC98] and [Bie95]) and for each* $K' \subseteq K$ *(in particular for* $K' = \emptyset$*) the functor* $U_{K,K'}$ *is logical w.r.t. the* $\mathbf{NCOH}_K$ *and* $\mathbf{NCOH}_{K'}$ *LL semantics.*

Proof. We equip $\mathbf{NCOH}_K$ with the comonad structure $(!, \mathrm{der}, \mathrm{dig})$ of **Rel**. We exploit the fact that the required commutative diagrams already hold in **Rel** and therefore also in $\mathbf{NCOH}_K$. Hence to check that $(!, \mathrm{der}, \mathrm{dig})$ is really a comonad we only need to prove that if f is a clique of $X \multimap Y$ then $!f$ is a clique of $!X \multimap !Y$, that der_X is a clique of $!X \multimap X$ and that dig_X is a clique of $!X \multimap !!X$. The same for the monoidality of the adjunction: we only need to ckeck that the **Rel** isomorphisms $!\top \cong 1$ and $!(X \& Y) \cong !X \otimes !Y$ are cliques (in both directions).

Let $[(x_j, y_j) \mid j \in J] \in \mathcal{M}_K(!f)$. If $[b_j \mid j \in J] \lhd [y_j \mid j \in J]$ then by construction of $!f$ there exists $[a_j \mid j \in J]$ such that $[(a_j, b_j) \mid j \in J] \in \mathcal{M}_K(f)$ and $[a_j \mid j \in J] \lhd [x_j \mid j \in J]$. Remark that since f is a clique, we have $[(a_j, b_j) \mid j \in J] \in \asymp_{X \multimap Y}$. In particular, if $[b_j \mid j \in J] \in \smile_Y$ then $[a_j \mid j \in J] \in \smile_X$. Hence if $[y_j \mid j \in J]$ admits a strict incoherent section then $(x_j)_{j \in J}$ admits one too. So

$$[y_j \mid j \in J] \in \smile_{!Y} \implies [x_j \mid j \in J] \in \smile_{!X}.$$

Now suppose $[x_j \mid j \in J] \in \asymp_{!X}$ and $[y_j \mid j \in J] \in \mathsf{N}_{!Y}$. We must prove that $[x_j \mid j \in J] \in \mathsf{N}_{!X}$. There exists $(b^i_j)_{(i,j) \in I \times J}$ such that

$$\forall j \in J, y_j = [b^i_j \mid i \in I] \text{ and } \forall i \in I, [b^i_j \mid j \in J] \in \mathsf{N}_Y.$$

By construction of $!f$ there exists $(a^i_j)_{(i,j) \in I \times J}$ such that

$$\forall (i,j) \in I \times J, (a^i_j, b^i_j) \in f \text{ and } \forall j \in J, x_j = [a^i_j \mid i \in I].$$

Since $[x_j \mid j \in J] \in \asymp_{!X}$ for each $i \in I$, $[a^i_j \mid j \in J] \in \asymp_X$. But, for each $i \in I$, $[(a^i_j, b^i_j) \mid j \in J] \in \mathcal{M}_K(f) \subseteq \asymp_{X \multimap Y}$ and $[b^i_j \mid j \in J] \in \mathsf{N}_Y$ so $[a^i_j \mid j \in J] \in \mathsf{N}_X$, for each $i \in I$. Finally $[x_j \mid j \in J] \in \mathsf{N}_{!X}$ which concludes the proof that $!f$ is a clique.

The fact that der_X is a clique is straightforward. We now prove that dig_X is a clique of $!X \multimap !!X$. Let $[(\sum_{i \in I_j} x^j_i, [x^j_i \mid i \in I_j]) \mid j \in J] \in \mathcal{M}_K(\mathrm{dig}_X)$.

Suppose $[[x^j_i \mid i \in I_j] \mid j \in J] \in \smile_{!!X}$. Then this multiset admits a section $[y_j \mid j \in J]$ strictly incoherent in $!X$. Hence this section $[y_j \mid j \in J]$ admits a section $[a_j \mid j \in J]$ strictly incoherent in X. Clearly this last section is also a section of $[\sum_{i \in I} x^j_i \mid j \in J]$ so this multiset is strictly incoherent in $!X$.

Now suppose $[\sum_{i \in I_j} x^j_i \mid j \in J] \in \asymp_{!X}$ and $[[x^j_i \mid i \in I_j] \mid j \in J] \in \mathsf{N}_{!!X}$. Then there exists a family $(y^j_i)^{j \in J}_{i \in I}$ such that: for all $j \in J$, $[y^j_i \mid i \in I]$ equals $[x^j_i \mid i \in I_j]$ (so $I = I_j$ and $\sum_{i \in I} y^j_i = \sum_{i \in I_j} x^j_i$); and for all $i \in I$, $[y^j_i \mid j \in J] \in \mathsf{N}_{!X}$. Hence for each $i \in I$, there exists a family $(a^j_{i,l})^{j \in J}_{l \in L_i}$ such that for all $j \in J$, $y^j_i = [a^j_{i,l} \mid l \in L_i]$ and such that for all $l \in L_i$, $[a^j_{i,l} \mid j \in J] \in \mathsf{N}_X$. Without any lost of generalities the L_i can be chosen pairwise disjoint. Setting $L = \cup_{i \in I} L_i$, we then have $\sum_{l \in L} a^j_l = \sum_{i \in I_j} x^j_i$ and for all $l \in L$, $[a^j_l \mid j \in J] \in \mathsf{N}_X$. Hence $[\sum_{i \in I_j} x^j_i \mid j \in J] \in \mathsf{N}_{!X}$.

The set $\{([], *)\}$ is a clique of $!\top \multimap 1$ and the set $\{(*, [])\}$ is a clique of $1 \multimap !\top$ so $!\top \cong 1$. We now prove that $!(X \& Y) \cong !X \otimes !Y$, for each X and Y. The graph f of the bijection map

$$\begin{cases} \mathcal{M}_{\mathrm{fin}}(|X|) \times \mathcal{M}_{\mathrm{fin}}(|Y|) & \to & \mathcal{M}_{\mathrm{fin}}(|X \& Y|) \\ (x, y) & \mapsto & x + y \end{cases}$$

is a relational isomorphism. It remains to prove that f is a clique of $!(X\&Y) \multimap (!X \otimes !Y)$ and that its transpose is a clique of $(!X \otimes !Y) \multimap !(X\&Y)$. Consider a multiset $[((x_i, y_i), x_i + y_i) \mid i \in I] \in \mathcal{M}_K(f)$. Since an element of $\smile_{X\&Y}$ is either an element of $\smile_X$ or an element of $\smile_Y$, a section s of $[x_i + y_i \mid i \in I]$ is strictly incoherent in $X\&Y$ iff s is a strictly incoherent section of $[x_i \mid i \in I]$ or of $[y_i \mid i \in I]$. It follows that

$$[x_i + y_i \mid i \in I] \in \smile_{!(X\&Y)} \iff [(x_i, y_i) \mid i \in I] \in \smile_{!X \otimes !Y}.$$

An element of $\mathsf{N}_{X\&Y}$ is either an element of N_X or an element N_Y. Hence, if $[x_i + y_i \mid i \in I]$ is neutral in $!(X\&Y)$, there exists a family $(c^j_i)^{j \in J}_{i \in I}$ such that for each $j \in J$, $[c^j_i \mid i \in I] \in \mathsf{N}_{X\&Y}$ and such that $J = J_X + J_Y$ with, for each $i \in I$, $[c^j_i \mid j \in J_X] = x_i$ and $[c^j_i \mid j \in J_Y] = y_i$ and this family splits into two families, the first one corresponding to the neutrality of $[x_i \mid i \in I]$ in $!X$ and the other one to the neutrality of $[y_i \mid i \in I]$ in $!Y$. Consequently the neutrality of $[x_i + y_i \mid i \in I]$ in $!(X\&Y)$ implies the neutrality of $[(x_i, y_i) \mid i \in I]$ in $!X \otimes !Y$. The converse is straightforward. So the required isomorphisms $!\top \cong 1$ and $!(X \,\&\, Y) \cong !X \otimes !Y$ holds. At last we obtain for free that this two isomorphisms are naturals and that the adjonction involved by the comonad is monoidal (see [Bie95]) just by using the fact that this is already the case in **Rel**. $\square$

4.4 The *of course* is the co-free commutative $\otimes$-comonoid

A commutative comonoid on a symmetric monoidal category $\mathcal{C}$, with respect to a monoidal structure $(\otimes, \mathrm{sym}, \mathrm{ass}, \mathrm{unit})$, is a 3-tuple $M = (\underline{M}, u_M, \mu_M)$, where $\underline{M} \in \mathcal{C}$, $u_M \in \mathcal{C}(\underline{M}, 1)$ and $\mu_M \in \mathcal{C}(\underline{M}, \underline{M} \otimes \underline{M})$, such that the following diagrams commute:

$$\begin{tikzcd}
\underline{M} \otimes \underline{M} \arrow[rr, "\mathrm{id}_{\underline{M}} \otimes \mu_M"] \arrow[drr, "\mu_M \otimes \mathrm{id}_{\underline{M}}"'] & & \underline{M} \otimes (\underline{M} \otimes \underline{M}) \arrow[d, "\mathrm{ass}_{\underline{M},\underline{M},\underline{M}}"] \\
& & (\underline{M} \otimes \underline{M}) \otimes \underline{M}
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{M} \arrow[r, "\mu_M"] \arrow[dr, "\mathrm{unit}_{\underline{M}}"'] & \underline{M} \otimes \underline{M} \arrow[d, "\mathrm{id}_{\underline{M}} \otimes u_M"] \\
& \underline{M} \otimes 1
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{M} \arrow[r, "\mu_M"] \arrow[dr, "\mu_M"'] & \underline{M} \otimes \underline{M} \arrow[d, "\mathrm{sym}_{\underline{M} \otimes \underline{M}}"] \\
& \underline{M} \otimes \underline{M}
\end{tikzcd}$$

associativity neutrality commutativity

A comonoid morphism f from $(\underline{M}, u_M, \mu_M)$ to $(\underline{N}, u_N, \mu_N)$ is a morphism $f \in \mathcal{C}(M, N)$ such that the following diagrams commute:

$$\begin{tikzcd}
\underline{M} \arrow[r, "f"] \arrow[dr, "u_M"'] & \underline{N} \arrow[d, "u_N"] \\
& 1
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{M} \arrow[r, "f"] \arrow[d, "\mu_M"'] & \underline{N} \arrow[d, "\mu_N"] \\
\underline{M} \otimes \underline{M} \arrow[r, "f \otimes f"] & \underline{N} \otimes \underline{N}
\end{tikzcd}$$

In each categorical semantics $\mathcal{C}$ of linear logic the "of course" naturally provides a commutative comonoid $(!X, \mathrm{weak}, \mathrm{cont})$ for each object X: weak_X is $!\top_X$ where $\top_X$ is the unique morphism of $\mathcal{C}(X, \top)$ and cont_X is $(!\langle \mathrm{id}_X, \mathrm{id}_X\rangle) \mathbin{⨟} e_X$ where $\langle \mathrm{id}_X, \mathrm{id}_X\rangle$ denotes the pairing of the identity with itself and where e_X is the isomorphism $!(X \mathbin{\&} X) \cong !X \otimes !X$. Moreover for each $f \in \mathcal{C}(X, Y)$, $!f$ is a $\otimes$-comonoid morphism between $(!X, \mathrm{weak}_X, \mathrm{cont}_X)$ and $(!Y, \mathrm{weak}_Y, \mathrm{cont}_Y)$.

In $\mathbf{NCOH}_K$, $\mathrm{weak}_X = \{([], *)\}$ and $\mathrm{cont}_X = \{(x_1 + x_2, (x_1, x_2)) \mid x_1, x_2 \in |!X|\}$.

A commutative comonoid (F, u_F, μ_F) is said to be co-free over an object X of $\mathcal{C}$ when there exists a morphism $d \in \mathcal{C}(F, X)$ such that for each commutative comonoid (A, u_A, μ_A), and for each $f \in \mathcal{C}(A, X)$ there exists a unique comonoid morphism f_* from (A, u_A, μ_A) to (F, u_F, μ_F) such that $f_* \mathbin{⨟} d = f$.

$$\begin{tikzcd}
(A, u_A, \mu_A) \arrow[r, "f_*"] \arrow[dr, "f"'] & (F, u_F, \mu_F) \arrow[d, "d"] \\
& X
\end{tikzcd}$$

By extension the "of course" ! is said to be the co-free commutative $\otimes$-comonoid or, for short, to be co-free, when for each commutative comonoid (A, u_A, μ_A), for each $X \in \mathcal{C}$ and for each $f \in \mathcal{C}(A, X)$ there exists a unique comonoid morphism $f_* : (A, u_A, \mu_A) \to (!X, \mathrm{weak}_X, \mathrm{cont}_X)$ such that

$$f_* \mathbin{⨟} \mathrm{der}_X = f.$$

Remarque. If ! is co-free then $f_* = \mathrm{id}_* \mathbin{⨟} !f$ where id is the identity morphism in $\mathcal{C}(A, A)$.

Lemme 3. *In* **Rel** *the exponential is co-free. Moreover if (A, u_A, μ_A) is a commutative $\otimes$-comonoid in* **Rel** *then $(a, x) \in (\mathrm{id}_A)_*$ iff if $(a_i)_{1 \le i \le n}$ is such that $[a_1, \ldots, a_n] = x$ then $\exists (b_i)_{0 \le i \le n}$ such that $b_0 = a$, $(b_i, (a_{i+1}, b_{i+1})) \in \mu_A$ for each $i < n$, and $(b_n, *) \in u_A$.*

Théorème 1 (co-free). *The "of course" ! is the co-free commutative $\otimes$-comonoid of $\mathbf{NCOH}_K$ and the forgetful functor $U_{K,\emptyset} : \mathbf{NCOH}_K \to \mathbf{Rel}$ maps this structure to the standard one.*

Proof. We prove that for each commutative comonoid (A, u_A, μ_A) of $\mathbf{NCOH}_K$ for each $X \in \mathbf{NCOH}_K$ and for each $f \in \mathbf{NCOH}_K(A, X)$, there exists a unique comonoid morphism $f_* : (A, u_A, \mu_A) \to (!X, \mathrm{der}, \mathrm{cont})$ such that $f_* \mathbin{⨟} \mathrm{der} = f$.

But if there is such an f_* in $\mathbf{NCOH}_K$ then $U_{K,\emptyset}(f_*)$ is a comonoid morphism from $(U_{K,\emptyset}(A), U_{K,\emptyset}(u_A), U_{K,\emptyset}(\mu_A))$ to $(U_{K,\emptyset}(!X), U_{K,\emptyset}(\mathrm{der}_X), U_{K,\emptyset}(\mathrm{cont}))$ and

$$U_{K,\emptyset}(f_*) \,\mathring{,}\, U_{K,\emptyset}(\mathrm{der}_X) = U_{K,\emptyset}(f).$$

As $(U_{K,\emptyset}(!X), U_{K,\emptyset}(\mathrm{der}_X), U_{K,\emptyset}(\mathrm{cont}))$ is the co-free $\otimes$-comonoid in $\mathbf{Rel}$ this means that, in $\mathbf{Rel}$,

$$U_{K,\emptyset}(f_*) = U_{K,\emptyset}(f)_*.$$

Moreover

$$U_{K,\emptyset}(f)_* = U_{K,\emptyset}(\mathrm{id})_* \,\mathring{,}\, !U_{K,\emptyset}(f)$$

and

$$!U_{K,\emptyset}(f) = U_{K,\emptyset}(!f).$$

So the only thing to prove is that $U_{K,\emptyset}(\mathrm{id})_*$ is a clique of $\mathrm{Cl}(A \multimap !A)$.

Let $[(a^i, [a^i_1, \ldots, a^i_{n_i}]) \mid i \in I]$ be an element of $\mathcal{M}_K(\mathrm{id}_*)$. Then, using Lemma 3, for each $i \in I$, let $(b^i_j)_{0 \leq j \leq n_i}$ be a family such that $b^i_0 = a^i$, $(b^i_j, (a^i_{j+1}, b^i_{j+1})) \in \mu_A$ for each $j < n_i$, and $(b^i_{n_i}, *) \in u_A$.

Suppose $[[a^i_1, \ldots, a^i_{n_i}] \mid i \in I] \in \smile_{!A}$ then this multiset admits a strict incoherent section. Up to a choice of an adequate indexation of the multiset $[a^i_1, \ldots, a^i_{n_i}]$, we can suppose without any loss of generality that this section is $[a^i_1 \mid i \in I]$. Remark that due to the existence of a section, none of the n_i is zero. We then have $[(a^i, (a^i_1, b^i_1)) \mid i \in I] \in \mathcal{M}_K(\mu_A)$ with $[a^i_1 \mid i \in I] \in \smile_A$. Hence $[(a^i_1, b^i_1) \mid i \in I] \in \smile_{A \otimes A}$. And, since $[(a^i, (a^i_1, b^i_1)) \mid i \in I]$ must be coherent for μ_A to be a clique of $A \multimap (A \otimes A)$, we then have $[a^i \mid i \in I] \in \smile_A$.

Now suppose $[a^i \mid i \in I] \in \mathrel{\overset{\frown}{\smile}}_A$ and $[[a^i_1, \ldots, a^i_{n_i}] \mid i \in I] \in \mathsf{N}_A$. According to the definition of neutrality in the "of course", all the n_i are equal, say $n_i = n (\forall i \in I)$, and, up to an appropriate re-indexing, $[a^i_j \mid i \in I] \in \mathsf{N}_A$, for each $1 \leq j \leq n$. Since $[(b^i_n, *) \mid i \in I] \in \mathcal{M}_K(u_A) \subseteq \mathrel{\overset{\frown}{\smile}}_{A \multimap 1}$ and $[* \mid i \in I] \in \mathsf{N}_1$, this means that $[b^i_n \mid i \in I] \in \asymp_A$. Now suppose $[b^i_{k+1} \mid i \in I] \in \asymp_A$ for a certain $k < n$, then using $[a^i_{k+1} \mid i \in I] \in \mathsf{N}_A$ and $[(b^i_k, (a^i_{k+1}, b^i_{k+1})) \mid i \in I] \in \mathcal{M}_K(\mu_A)$ it follows that $[b^i_k \mid i \in I] \in \asymp_A$. Thus, for all $j \leq n$, $[b^i_j \mid i \in I] \in \asymp_A$ and in particular $[b^i_0 \mid i \in I] = [a^i \mid i \in I]$ is then proved to be both coherent and incoherent, that is to be neutral. So id_* is a clique. $\square$

Consider a sub-category $\mathcal{C}$ of $\mathbf{NCOH}_K$ which is a categorical semantics of intuitionistic multiplicative exponential linear logic[3]. Let E be the operation modeling the "of course" on objects in $\mathcal{C}$. We shall say that this semantics is *multiset based* if for each $X \in \mathcal{C}$:

- the web of $E(X)$ is made of multisets of points of the web of X (*i.e.* $|E(X)| \subseteq \mathcal{M}_{\mathrm{fin}}(|X|)$);
- the commutative comonoid structure provided with $E(X)$ by the semantics is defined by $\mathrm{weak}'_X = \{([], *)\}$ (of type $E(X) \to 1$) and

  $$\mathrm{cont}'_X = \{(x_1 + x_2, (x_1, x_2)) \mid x_1 + x_2 \in |E(X)| \text{ et } x_1, x_2 \in \mathcal{M}_{\mathrm{fin}}(|X|)\}$$

  (of type $E(X) \to E(X) \otimes E(X)$);
- the associated dereliction morphism is

  $$\mathrm{der}'_X = \{([a], a) \mid a \in |X|\} \text{ (of type } E(X) \to X).$$

[3]We do not require $\mathcal{C}$ satisfies more, but a typical $\mathcal{C}$ for our purpose will be a new Seely category where the multiplicative additive and orthogonal constructions are the ones of $\mathbf{NCOH}_K$ and so one should have a semantics for the full linear logic fragment, where the exponentials are given by a comonad.

Corollaire 1 (maximality of the co-free "of course"). *If a sub-monoidal category $\mathcal{C}$ of $\mathbf{NCOH}_K$ is a multiset based LL semantics, of "of course" E then, for each object $X \in \mathcal{C}$,*

$$\smile_{E(X)} \subseteq \smile_{!X} \tag{10}$$

et

$$\frown_{E(X)} \subseteq \frown_{!X}. \tag{11}$$

"Sub-monoidal category" means that $\mathcal{C}$ is a sub-category of $\mathbf{NCOH}_K$ equipped with the same symmetric monoidal structure as $\mathbf{NCOH}_K$.

Proof. Since $\mathcal{C}$ is a semantics of linear logic, $E(X)$ comes with a $\otimes$-comonoid structure $(E(X), \mathrm{weak}'_X, \mathrm{cont}'_X)$ where weak'_X is the weakening morphism and cont'_X is the contraction morphism. Let der'_X be the dereliction morphism for X of $\mathcal{C}$. Using Theorem 1, there exists a morphism $\mathrm{der}'_{X,*}$ of $E(X) \multimap\, !X$. Using Lemma 3 and due to the fact that $\mathcal{C}$ is multiset based we obtain that $\mathrm{der}'_{X,*}$ is equal to $\{(x,x) \mid x \in |E(X)|\}$ (the inclusion morphism of $E(X)$ in $!X$). Finally, using Equation (5), it yields Equation (10) and Equation (11). $\square$

We shall say that a multiset based semantics of LL in a sub-category $\mathcal{C}$ of $\mathbf{NCOH}_K$ is *non uniform* when the web of the "of course" E is the whole set of finite multisets (*i.e.* $|E(X)| = \mathcal{M}_{\mathrm{fin}}(|X|), \forall X \in \mathcal{C}$).

Corollaire 2 (sequentiality failure).
Each non uniform multiset based semantics of LL in a sub-monoidal category $\mathcal{C}$ of $\mathbf{NCOH}_K$ fails to reject the morphism $\{([v], v), ([f], v), ([v, f], v)\}$ of type ***bool*** $\rightarrow$ ***bool****.*

Proof. One easily verifies that $\mathcal{M}_{\mathbb{N}\setminus\{0,1\}}(\{[\mathtt{v}], [\mathtt{f}], [\mathtt{v}, \mathtt{f}]\}) \cap \frown_{!\mathbf{bool}} = \emptyset$ thus the morphism above is indeed accepted by the $\mathbf{NCOH}_{\mathbb{N}\setminus\{0,1\}}$ semantics. So this set is *a fortiori* a morphism in each $\mathbf{NCOH}_K$ semantics and, by maximality of the "of course", in any multiset based non uniform semantics in a sub-monoidal category of $\mathbf{NCOH}_K$. $\square$

This is a strong negative results since this set cannot be included in the interpretation of a term of PCF. Our sentiment is that it will be the same for any reasonably sequential calculus interpretable in our semantics of linear logic. Remark that the very similar morphism $\{([\mathtt{v}, \mathtt{v}], \mathtt{v}), ([\mathtt{f}, \mathtt{f}], \mathtt{v}), ([\mathtt{v}, \mathtt{f}], \mathtt{v})\}$ is the interpretation of $\lambda b.\, \mathtt{if}\, b\, \mathtt{then}\, (\mathtt{if}\, b\, \mathtt{then}\ \mathtt{v}\ \mathtt{else}\ \mathtt{v})\ \mathtt{else}\ (\mathtt{if}\, b\, \mathtt{then}\ \mathtt{v}\ \mathtt{else}\ \mathtt{v})$.

4.5 Determinism

From now on, we consider that K is a *non-empty* subset of $\mathbb{N}\setminus\{0,1\}$. In that case the power $\mathcal{M}_K$ is strictly monotone and preserves disjointness.

Définition 3. Let $\mathbf{NCoh}_K$ be the full sub-category of $\mathbf{NCOH}_K$ whose objects are the weakly reflexive K-cohérent spaces.

Let us recall that being weakly reflexive for a K-cohérent space X means that:

$$\mathsf{N}_X \subseteq \cup_{a \in |X|} \mathcal{M}_K(\{a\}). \tag{12}$$

Clearly $\mathbf{NCoh}_K$ is closed under the orthogonal, additive and multiplicative constructions. This is also the case for the exponential construction as easily verified. Indeed assume X is weakly reflexive and consider a neutral multiset $[x_i \mid i \in I]$ in $!X$. Then there exists a family $(a_i^j)_{i \in I}^{1 \le j \le p}$ such that, for each $i \in I$, $x_i = [a_i^j \mid 1 \le j \le p]$ and, for each $1 \le j \le p$, $[a_i^j \mid i \in I] \in \mathsf{N}_X$. So using weak reflexivity of X we obtain that there exists a family $(a^j)_{1 \le j \le p}$ such that $a_i^j = a^j (\forall i, j)$ and consequently all the x_i are equal.

Hence this sub-category is a denotational semantics of propositional linear logic. Each forgetful functor $U_{K,K'}$ between $\mathbf{NCOH}_K$ and $\mathbf{NCOH}_{K'}$ (for $K' \subseteq K$) defines a forgetful functor between $\mathbf{NCoh}_K$ and $\mathbf{NCoh}_{K'}$ having similar properties and for which we use the same notation $U_{K,K'}$.

Proposition 4 (determinism). *If $X \in \mathbf{NCoh}_K$ and if x is a clique of X and y is an anti-clique of X (that is a clique of $X^\perp$) then $\sharp(x \cap y) \leq 1$.*

This is a direct consequence of Proposition 1.

Remarque. It is worth remarking that in non-uniform K-cohérent semantics we do not have the second part of Property 2: if $(a, c) \in f \,;\, g$ for $f : X \to Y$ and $g : Y \to Z$ then there exists a b such that $(a, b) \in f$ and $(b, c) \in g$ but b is not necessarily unique. But uniqueness of b holds again if $\mathcal{M}_K(\{(a, c)\}) \subseteq \mathsf{N}_{X\to Z}$ and moreover in that case b is such that $\mathcal{M}_K(\{b\}) \subseteq \mathsf{N}_Y$.

We are not more interested in non weakly reflexive K-cohérent spaces, and the category $\mathbf{NCOH}_K$. In the sequel, the most general category will be $\mathbf{NCoh}_K$.

5 Relating uniform and non uniform semantics

We now intend to define uniform K-cohérent semantics and to relate them with non uniform K-cohérent semantics.

5.1 The neutral web

Proposition 4 can be made more precise since only certain points can be at the intersection of a clique and an anti-clique. These points constitute the *neutral web*.

Définition 1. Let $X \in \mathbf{NCoh}_K$. We call *neutral web* of X and we denote by $|X|_{\mathsf{N},K}$ (or simply by $|X|_{\mathsf{N}}$) the set $\{a \in |X| \mid \mathcal{M}_K(\{a\}) \subseteq \mathsf{N}_X\}$.

Exemple 1. For the K-cohérent space G of Example 2 page 19 we have: $[a, b] \in |!G|_{\mathsf{N},K}$, if $K \subseteq \{2\}$, $[a, b, c] \in |!G|_{\mathsf{N},K}$ and elsewhere $[a, b, c] \notin |!G|_{\mathsf{N},K}$.

A key result about the neutral web is its behaviour when an "of course" construction is performed:

Lemme 1 (key lemma). *For $X \in \mathbf{NCoh}_K$ one has*

$$|!X|_{\mathsf{N},K} = \{x \in \mathcal{M}_{fin}(|X|_{\mathsf{N},K}) \mid \mathrm{supp}(x) \in \mathrm{Cl}(X)\}$$

Proof. Let $x \in |!X|_{\mathsf{N},K}$. Then for all $k \in K$, there exists a family $(a_i^j)_{1\leq i\leq k}^{j\in J}$ such that $[a_i^j \mid j \in J] = x$ and $[a_i^j \mid 1 \leq i \leq k] \in \mathsf{N}_X$. Due to Equation (12), for each $j \in J$, $a_1^j = \ldots = a_k^j$. Hence for all $k \in K$, for all $a \in x$, $k.[a] \in \mathsf{N}_X$. So $\mathrm{supp}(x) \subseteq |X|_{\mathsf{N},K}$. Each $y \in \mathcal{M}_K(\mathrm{supp}(x))$ is a section of the multiset $(\sharp y).[x] \in \mathsf{N}_{!X} \subseteq \frown_{!X}$, hence $\mathrm{supp}(x)$ is a clique. Thus the left to right inclusion is proved. Conversely, let $x \in \mathcal{M}_{\mathrm{fin}}(|X|_{\mathsf{N},K})$. If $\mathrm{supp}(x)$ is a clique then $k.[x] \in \frown_{!X}$ for any $k \in K$. Moreover each of the element a of x satisfies $k.[a] \in \mathsf{N}_X$ thus $k.[x] \in \frown_{!X}$ for any $k \in K$. And this proves the right to left inclusion. $\square$

Exemple 2. In $(!G)^\perp$, the set $x = \{[a, b], [a, c]\} \subseteq |(!G)^\perp|_{\mathsf{N},K}$ is not a clique if $2 \in K$ but is a clique if $2 \notin K$. Hence $[[a, b], [a, c]] \notin |!(!G)^\perp|_{\mathsf{N},\{2\}}$ and $[[a, b], [a, c]] \in |!(!G)^\perp|_{\mathsf{N},\{3\}}$.

The property stated in this lemma teach us that a restriction to the reflexive subspaces of K-cohérent spaces has good chances to provide us with a new version of the semantics comparable to the multiset based coherence space semantics, Equation (4), when $K = \{2\}$. This will be successfully shown, among others things, in the next section. A more direct consequence, Proposition 2, is that such a restriction can be performed at any inductive step of the interpretation. Provided it is performed at the last step, the resulting reflexive object (in the case of the interpretation of a formula) or morphism (in the case of the interpretation of a proof) will be the same.

Définition 2. If $X \in \mathbf{NCoh}_K$, the *neutral restriction* of X is the *sub-space* of X of web $|X|_{\mathsf{N}}$, that is $(|X|_{\mathsf{N}}, \mathsf{N}_X \cap M, \frown_X \cap M, \smile_X \cap M)$ where $M = \mathcal{M}_K(|X|_{\mathsf{N}})$, and the neutral restriction of a clique x of X is $x \cap |X|_{\mathsf{N}}$. The functor $N_K : \mathbf{NCoh}_K \to \mathbf{NCoh}_K$, sometimes simply denoted by N, associates to objects and morphisms their neutral restrictions.

One easily verifies that N_K is indeed a functor.

Remarque. The K-cohérent space $|X|_{\mathsf{N}}$ is reflexive. Moreover it is the maximal reflexive subspace of X.

Proposition 2. *The functor N_K commutes with all the multiplicative additive constructions. Moreover $N_K! = N_K!N_K$.*

Proof. The first statement is an obvious consequence of the corresponding definitions.

On objects, $N_K! = N_K!N_K$ is a consequence of Lemma 1. Indeed, in the right part of the equality stated in this lemma, $\mathrm{Cl}(X)$ can be replaced with $\mathrm{Cl}(N_K X)$ since $\mathrm{supp}(x) \subseteq |X|_{\mathsf{N},K}$. This gives $|!X|_{\mathsf{N},K} = |!N_K X|_{\mathsf{N},K}$ which is what we wanted. The equality $N_K! = N_K!N_K$ on morphism is a straightforward consequence of the equality on objects. □

5.2 Uniform K-coherence semantics

In this section, we define a uniform K-coherence semantics in the full sub-category of $\mathbf{NCoh}_K$ which objects are reflexive K-cohérent spaces. So uniform will be a synonym of reflexive for objects of $\mathbf{NCoh}_K$.

We denote by $\mathbf{Coh}_K$ the full sub-category of $\mathbf{NCoh}_K$ whose objects are the K-cohérent uniforme spaces.

A $\{2\}$-cohérent uniforme space is just an ordinary coherence space.

The functor N_K maps $\mathbf{NCoh}_K$ to $\mathbf{Coh}_K$ and on $\mathbf{Coh}_K$, N_K acts like the identity functor.

Additive and multiplicative constructions of $\mathbf{NCoh}_K$ preserve uniform K-cohérent spaces. This is not the case for the "of course" functor. Fortunately, Lemma 1 gives a clear hint on what should be the right exponentials for $\mathbf{Coh}_K$.

Définition 3. We define the functor $\underset{u}{!}$ interpreting the "of course" in $\mathbf{Coh}_K$ by setting $\underset{u}{!} = N_K!$. We denote by $\underset{u}{?}$ the corresponding "why not" functor.

The web of $\underset{u}{!}\,X$, called the *uniform web*, is then

$$|\underset{u}{!}\,X| = \{x \in \mathcal{M}_{\mathrm{fin}}(|X|) \mid \mathrm{supp}(X) \in \mathrm{Cl}(X)\}$$

and the coherence of $\underset{u}{!}\,X$ is then given by

$$M \in \frown_{\underset{u}{!}\,X} \text{ ssi } \{m \mid m \lhd M\} \subseteq \frown_X .$$

This definition of the exponentials appears as a multiplicities aware version of the hypercoherences exponentials that have been introduced in [Ehr93].

As stated by the following theorem, these definitions give rise to a new class of uniform semantics together with a straightforward way to extract these interpretations from the non uniform ones.

Théorème 1. *For each $K \subseteq \mathbb{N}\backslash\{0,1\}$, $\mathbf{Coh}_K$ equipped with the uniform exponentials and the standard multiplicative additive structures of $\mathbf{NCoh}_K$ is a categorical semantics of linear logic. Moreover:*

1. *the functor $N_K : \mathbf{NCoh}_K \to \mathbf{Coh}_K$ is* logical *which means in particular that the* neutral restriction *of the K-cohérent space $[A]_K$ is the uniform K-coherent interpretation $[A]^u_K$ of a formula A and that the neutral restriction $[\pi]_K \cap |\vdash \Gamma|_{\mathsf{N},K}$ of the K-cohérent interpretation of a proof π of a sequent $\vdash \Gamma$ is the uniform K-cohérent interpretation $[\pi]^u_K$ of π;*

2. *when $K = \{2\}$ this semantics is exactly the usual multiset based coherence semantics.*

Proof. The multiplicative-additive part of the verification of the fact that $\mathbf{Coh}_K$ is a semantics of linear logic is easy and relies essentially on the fact that N commutes to all the additive and multiplicative constructions.

The exponential part is not very complicated either. By setting $\mathrm{der}_{u,X} = N(\mathrm{der}_X)$ and $\mathrm{dig}_{u,X} = N(\mathrm{dig}_X)$ for each $X \in \mathbf{Coh}_K$, we obtain two natural transformations $\mathrm{der}_u : N! \dot{\to} N\,\mathrm{id}$ and $\mathrm{dig}_u : N! \dot{\to} N!!$ in $\mathbf{Coh}_K$.

But N is the identity functor on $\mathbf{Coh}_K$, $N! = \underset{u}{!}$ and using Proposition 2 ($N! = N!N$ in $\mathbf{NCoh}_K$) we obtain $N!! = \underset{u}{!}\,\underset{u}{!}$, and also $N!!! = \underset{u}{!}\,\underset{u}{!}\,\underset{u}{!}$. So der_u and dig_u are in fact natural transformations $\mathrm{der}_u : \underset{u}{!} \dot{\to} \mathrm{id}$ and $\mathrm{dig}_u : \underset{u}{!} \dot{\to} \underset{u}{!}\,\underset{u}{!}$.

These two natural transformations endow $\underset{u}{!}$ with a comonad structure. In fact we deduce the commutation of the required diagrams from the commutation of the corresponding diagrams already holding for the comonad $(!, \mathrm{der}, \mathrm{dig})$ by use of the functor N. The only non-obvious step is then to prove that for each $X \in \mathbf{Coh}_K$,

$$N\,\mathrm{dig}_{!X} = \mathrm{dig}_{u,\underset{u}{!}X} \text{ and } N\,\mathrm{der}_{!X} = \mathrm{der}_{u,\underset{u}{!}X}\,.$$

This can be done as follows. For all $f \in \mathbf{NCoh}_K(N!X, !X)$ one has

$$!f \mathbin{;} \mathrm{dig}_{!X} = \mathrm{dig}_{N!X} \mathbin{;} f$$

hence

$$N(!f \mathbin{;} \mathrm{dig}_{!X}) = N(\mathrm{dig}_{!N!X} \mathbin{;} f)$$

and so

$$N!f \mathbin{;} N\,\mathrm{dig}_{!X} = N(\mathrm{dig}_{N!X}) \mathbin{;} Nf. \tag{13}$$

The set $\mathrm{id}_{N!X}$ is clearly a clique of $N!X \multimap !X$ and so it can be seen as a (inclusion) morphism i from $N!X$ to $!X$. We then have the set equalities

$$N!i = \mathrm{id}_{\underset{u}{!}\,\underset{u}{!}X} \text{ and } Ni = \mathrm{id}_{\underset{u}{!}X}$$

so finally by taking $f = i$ in Equation (13) we obtain the set equality

$$N\,\mathrm{dig}_{!X} = N\,\mathrm{dig}_{N!X} \text{ that is } N\,\mathrm{dig}_{!X} = \mathrm{dig}_{u,\underset{u}{!}X}\,.$$

Starting from the equation

$$N(\mathrm{der}_{!X} \mathbin{;} i) = N(!i \mathbin{;} \mathrm{der}_{!X})$$

one proves

$$N\,\mathrm{der}_{!X} = \mathrm{der}_{u,\underset{u}{!}X}$$

in the same way.

Using Proposition 2 we obtain the isomorphisms $\underset{u}{!}A \otimes \underset{u}{!}B \cong \underset{u}{!}(A\&B)$ and $\underset{u}{!}\top \cong 1$.

$\mathbf{Coh}_K$ has been proved to be a categorical semantics of linear logic and there is nothing more to say for stating that N is logical.

The comonoid structure of the exponential $\underset{u}{!}$ is then the image of the comonoid structure of the exponential $!$ of $\mathbf{NCoh}_K$ through the functor N. The fact that $(\underset{u}{!}, \mathrm{der}_u)$ is co-free relies essentially on the set equality $\mathrm{der}_{u,X} = \mathrm{der}_X$ ($\forall X \in \mathbf{Coh}_K$) which is just a consequence of the fact that all singletons are cliques in $\mathbf{Coh}_K$. In fact, given a commutative comonoid (A, u_A, μ_A) of $\mathbf{Coh}_K$ and $f \in \mathbf{Coh}_K(A, X)$ one has

$$N(f_*) \mathbin{;} \mathrm{der}_{u,X} = N(f_* \mathbin{;} \mathrm{der}_X)$$

for each $f \in \mathbf{Coh}_K(A, X)$ where f_* is the unique comonoid morphism $A \to !X$ such that $f_* \mathbin{;} \mathrm{der}_X = f$. But $N(f_*) : A \to \underset{u}{!}X$ is also a comonoid morphism. Remark that the inclusion

morphism $i : \underset{u}{!} X \to !X$ is a comonoid morphism hence $N(f_*) \mathbin{;} i : A \to !X$ is a comonoid morphism. We also have the set equalities

$$N(f_*) \mathbin{;} i = N(f_*)$$

and, due to $\mathrm{der}_X = \mathrm{der}_{u,X}$,

$$N(f_*) \mathbin{;} i \mathbin{;} \mathrm{der}_X = N(f_*) \mathbin{;} \mathrm{der}_{u,X} = f.$$

By uniqueness of f_*, $N(f_*) \mathbin{;} i$ equals f_*, so we finally obtain the set equality $f_* = N(f_*)$, and the co-freeness of $\underset{u}{!}$ follows.

Finally, $[x, y] \in \frown_{\underset{u}{!} X}$ iff $\forall a \in x, \forall b \in y, [a, b] \in \frown_X$ that is, in $\mathbf{Coh}_{\{2\}}$, iff $\mathrm{supp}(x + y)$ is a clique. So in $\mathbf{Coh}_{\{2\}}$ which is the category of coherence spaces, $\underset{u}{!}$ is the well-known multiset based exponential of coherence spaces. □

Spelling out the categorical definition of the semantics, the interpretation of linear logic in $\mathbf{Coh}_K$ is now defined as its interpretation in **Rel** for the multiplicative-additive and identity groups and with an exponential group similarly defined but using uniform exponentials and the restriction they induce on the interpretation of proofs.

The promotion and the contraction rules cases are subject to the standard restrictions: in the case of the contraction take only the $(\gamma, \mu_1+\mu_2)$ such that $\mathrm{supp}(\mu_1+\mu_2)$ is a clique of $[A]_K{}^\perp$ and for the promotion, in $f^\dagger$, take only the points such that, for each $i \leq k$, $\mathrm{supp}(\sum_{j\in J} \mu_i^j) \in \mathrm{Cl}([A_i]_K{}^\perp)$. As for usual coherence spaces and hypercoherences, this condition is sufficient to ensure that $[a_1, \ldots, a_k] \in \mathrm{Cl}([A]_K)$ (under the assumption that f is truly a clique).

5.3 Multicoherences

We call the categorical semantics based on $\mathbf{Coh}_{\mathbb{N}\setminus\{0,1\}}$ the *multicoherence semantics*[4], we call *multicoherences* its objects, and we also call *non uniform multicoherences* the objects of $\mathbf{NCoh}_{\mathbb{N}\setminus\{0,1\}}$. The only difference between hypercoherences and multicoherences is that multicoherences take into account the multiplicity of points for the coherence relation.

Proposition 3 (sequentiality). *In the multicoherence semantics, every finite clique of function type* $!(\boldsymbol{bool} \,\&\, \ldots \,\&\, \boldsymbol{bool}) \multimap \boldsymbol{bool}$ *is sub-definable in* PCF.

Proof. The proof follows the same scheme as for the usual hypercoherence semantics. □

Remarque. All cliques in the multicoherence semantics are cliques in the coherence semantics (this is a consequence of the foliation property).

5.4 Non uniform hypercoherences

Hypercoherences can be seen as particular multicoherences: the multicoherences X such that

$$\forall u \in \frown_X, \mathrm{supp}^{-1}(\mathrm{supp}(u)) \subseteq \frown_X.$$

If X is a non uniform multicoherence having this property for both the coherence relation and the incoherence relation we say that X is a *non uniform hypercoherence*. So a non uniform hypercoherence is indeed simply a weakly reflexive $\mathcal{P}^*_{\mathrm{fin}}$-space. But it is more convenient here to present non uniform hypercoherences as particular non uniform multicoherences.

If X is a non uniform multicoherence, $S(X)$ is the non uniform hypercoherence defined by

$$\frown_{S(X)} = \{u \in \mathcal{M}_{\mathbb{N}\setminus\{0,1\}}(|X|) \mid \mathrm{supp}^{-1}(\mathrm{supp}(u)) \subseteq \frown_X\}$$
$$\mathsf{N}_{S(X)} = \{u \in \mathcal{M}_{\mathbb{N}\setminus\{0,1\}}(|X|) \mid \mathrm{supp}^{-1}(\mathrm{supp}(u)) \subseteq \mathsf{N}_X\}$$

[4] General graph theory misses a term for such graphs and, contrarily to the *hyper-* situation where hypercoherences and hypergraphs are the same, *multigraphs* already exist but are not multicoherences.

Remark that the operation $S\underset{u}{!}$ which maps X to $S(\underset{u}{!}X)$ is the hypercoherence multiset based exponential construction on objects.

Théorème 2.

1. *The sub-category* **NHc** *of* $\mathbf{NCoh}_{\mathbb{N}\setminus\{0,1\}}$ *of objects the non uniform hypercoherences, equipped with the exponential* $S!$ *on objects and acting like* $!$ *on morphisms is a semantics of linear logic.*

2. *The functor* N *from* **NHc** *to* **Hc**, *the category of hypercoherences, is logical (for the multiset based hypercoherence semantics).*

3. *The exponentials* $S!$ *and* $S\underset{u}{!}$ *are respectively co-free in* **NHc** *and* **Hc**.

Proof. The proof of these statements follows from the proofs of Proposition 2, Theorem 1 and Theorem 1. Just remark that some results can be re-used since for each non uniform multicoherence X, one has $S(!X) = S(!S(X))$ and $N(S(X)) = S(N(X))$. $\square$

Remark that Corollary 1 applies to $S!$ and $S\underset{u}{!}$.

Exemple 3. The K-cohérent space G of the Example 2 page 19 is uniform. And when $K = \mathbb{N}\setminus\{0,1\}$, G is an hypercoherence. The multisets $[a,b]$ and $[a,c]$ are elements of $|!G|_{\mathsf{N}}$. The set $x = \{[a,b],[a,c]\} \subseteq |!G|_{\mathsf{N}}$ is a anti-clique of $S(\underset{u}{!}G)$. But this set is not an anti-clique (nor a clique) of $\underset{u}{!}G$. Hence each finite multiset of support x is an element of $|?S(\underset{u}{!}G)|_{\mathsf{N}}$ but not an element of $|?\underset{u}{!}G|_{\mathsf{N}} = |?!G|_{\mathsf{N}}$.

For sake of direct usability, we spell out the definition of the non uniform exponential of hypercoherences on objects directly in the $\mathcal{P}^*_{\mathrm{fin}}$-space setting. The definition of this exponential, denoted $\underset{nuh}{!}$ is as follows.

If X is a weakly reflexive $\mathcal{P}^*_{\mathrm{fin}}$-space then $\underset{nuh}{!}X$ is the $\mathcal{P}^*_{\mathrm{fin}}$-space of web $|\underset{nuh}{!}X| = \mathcal{M}_{\mathrm{fin}}(|X|)$ and such that for each $x \subseteq^*_{\mathrm{fin}} |\underset{nuh}{!}X|$

$$\smile_{\underset{nuh}{!}X} \text{ ssi } \exists s \in \smile_X, s \lhd x \tag{14}$$

$$x \in \mathsf{N}_{\underset{nuh}{!}X} \text{ ssi } \exists \mu, x = \{\mu\} \text{ et } \forall a \in \mu, \{a\} \in \mathsf{N}_X \tag{15}$$

Of course, $\underset{nuh}{!}X$ is weakly reflexive.

5.5 Extensional collapses

Consider the situation where a same symmetric monoidal closed category has two different exponentials defining two different semantics of linear logic.

P.-A. Melliès has shown that if there is a *coercion* between the two exponentials which preserves some structure then the two semantics will have the same extensional collapse ([Mel04]). He uses this result to prove that the extensional collapse of the multiset-based hypercoherence semantics is the set-based hypercoherence semantics and he also reproved the same thing for the coherence spaces semantics (this was already proved by Barreiro and Ehrhard in [BE97]).

This result easily applies to our situation. We then obtain that the multiset based coherence semantics and the non uniform coherence space semantics have the same extensional collapse which is the set based coherence space semantics; the same for hypercoherences; and the same for multicoherences which we equip with the set based exponential $\underset{e}{!}$ defined by:

$$|\underset{e}{!}X| = \{x \in \mathcal{P}_{\mathrm{fin}}(|X|) \mid x \in \mathrm{Cl}(X)\}$$

$$M \in \frown_{\underset{e}{!}X} \text{ ssi } \{m \mid m \lhd M\} \subseteq \frown_X.$$

Of course this last exponential also provides a semantics of linear logic.

We can characterize more precisely the relation between extensional collapses of uniform and non uniform semantics.

Let $\mathcal{M}$ and $\mathcal{M}'$ be respectively the non uniform and the uniform semantics either of coherence spaces, hypercoherence or multicoherence semantics. Let $\approx$ and $\sim$ be the extensional PERs respectively on $\mathcal{M}$ and $\mathcal{M}'$. In what follows N is the neutral restriction functor.

Lemme 4. *Let σ and τ be simple types. If f is a clique of $\mathcal{M}(\sigma \to \tau)$ and x is a clique of $\mathcal{M}(\sigma)$ then the clique $N(f(x))$ of $\mathcal{M}'(\tau)$ is equal to $N(f)(N(x))$ and also to $N(f)(x)$.*

Proof. The equality $N(f)(N(x)) = N(f)(x)$ is trivial. We only prove $N(f(x)) = N(f)(N(x))$. Since $Nf \subset f$ and $Nx \subset x$, $N(f)(N(x)) \subset f(x)$ and since N is monotone $N(f)(N(x)) = N(N(f)(N(x))) \subset N(f(x))$. Conversely let $b \in N(f(x))$ then there exists a $\mu \in \mathcal{M}_{\text{fin}}(x)$ such that $(\mu, b) \in f$ and $k[b]$ is neutral for all $k \in K$ ($K = \{2\}$ or $K = \mathbb{N}\backslash\{0,1\}$). Since $k[b]$ is neutral and f is a clique $k[\mu]$ is incoherent in $!\mathcal{M}(\sigma)$ but since x is a clique $k[\mu]$ is also coherent in $!\mathcal{M}(\sigma)$. Thus $\mu \in N!\mathcal{M}(\sigma)$ and so $(\mu, b) \in Nf$, and $\mu \in \mathcal{M}_{\text{fin}}(Nx)$. This concludes by stating $b \in N(f)(N(x))$. $\square$

Lemme 5. *If σ is a simple type and if f and g are cliques of $\mathcal{M}(\sigma)$ then*

$$f \approx_\sigma g \text{ ssi } Nf \sim_\sigma Ng.$$

Proof. This is trivially true on basis type (on basis type N acts as the identity functor). Suppose this is true for types σ and τ. We prove the property for the type $\sigma \to \tau$. Let $f \approx_{\sigma\to\tau} g$ and let $x \sim_\sigma y$. Since $Nx = x$ and $Ny = y$, by induction hypothesis, $x \approx_\sigma y$. Hence $f(x) \approx_\tau g(y)$ and so, by induction hypothesis, $(Nf)(x) \sim_\tau (Ng)(y)$. So $f \approx_{\sigma\to\tau} g$ implies $Nf \sim_{\sigma\to\tau} Ng$. Conversely let f and g be two cliques of $\mathcal{M}(\sigma \to \tau)$ such that $Nf \sim_{\sigma\to\tau} Ng$. Let $x \approx_\sigma y$. Then, by induction hypothesis, $Nx \sim_\sigma Ny$. Hence $Nf(Nx) \sim_\tau Ng(Ny)$ and $N(f(x)) \sim_\tau N(g(y))$. And, by induction hypothesis, $f(x) \approx_\tau g(y)$. This concludes proving the lemma by stating that $Nf \sim_{\sigma\to\tau} Ng$ implies $f \approx_{\sigma\to\tau} g$. $\square$

Théorème 3. *The neutral functor N defines a one to one correspondence between the extensional collapse of $\mathcal{M}$ and the extensional collapse of $\mathcal{M}'$.*

This is a direct consequence of the last lemmas.

Finally using the fact that the set based hypercoherence and multicoherence semantics are both extensional we show that hypercoherences and multicoherences are extensionally different by exhibiting, in Example 4, a relation at a functional type which is a clique in one of two semantics but not in the other (this example was originally designed to exhibit a set which is a clique in the set-based hypercoherence semantics but not in the coherence spaces semantics).

Exemple 4. For the hypercoherence G of our last examples above, one has that $\{a, b\}$ and $\{c\}$ are elements of $|\underset{e}{!}G| = |S(\underset{e}{!}G)|$. Moreover $(\{a, b\}, \mathtt{v})$ and $(\{c\}, \mathtt{f})$ are elements of $|\underset{e}{!}G \multimap \mathbf{bool}| = |S(\underset{e}{!}G) \multimap \mathbf{bool}|$. The relation $F = \{(\{a, b\}, \mathtt{v}), (\{c\}, \mathtt{f})\}$ is a clique of the hypercoherence $S(\underset{e}{!}G) \multimap \mathbf{bool}$. But F is not a clique of the multicoherence $\underset{e}{!}G \multimap \mathbf{bool}$, since $[\{a, b\}, \{c\}]$ is coherent but $[\mathtt{v}, \mathtt{f}]$ is strictly incoherent.

6 Conclusion and further works

6.1 Static interactivity

The present work gives some strong evidence that static uniformity is a matter of restriction to possible results of computation, through interactions and especially through interactions in a closed case: between a *proof* of A and a *proof* of $A^\perp$. We further argue on this point by adopting basic ideas of the interactive point of view on computation developped in Girard's ludics [Gir01].

The main idea is to consider the linear logic system extended with new rules (such that the system still enjoys cut-elimination). Then a formula A and its linear negation $A^\perp$ can be both provable. Hence we can provoke interactions through cut elimination between proofs of A and proofs of $A^\perp$. If we further require that the extended logical system admits one of the deterministic non uniform semantics we presented then the consequence of determinism is that such an interaction involves at most one point.

Let suppose that we add two para-rules to linear logic:

$$\frac{}{\vdash}\,(abandon) \qquad \frac{}{\vdash\Gamma}\,(divergence)$$

a *give up* (this *para*-rule roughly corresponds to the *daemon* of Girard's Ludics [Gir01]) rule and a *divergence* rule whose respective interpretations in the relational semantics are a singleton (the unique point in the unit context $\perp$) and the empty set. It is easy to extend the cuts elimination procedure for this two rules and to check that for instance the relational semantics extends into a semantics of the strongly normalizing calculus we then obtain. In this setting a formula A and its dual $A^\perp$ are always both provable.

When we apply a cut rule between a proof π' of A and a proof π'' of $A^\perp$, the proof π we obtain normalizes into a proof of the empty sequent. And there is only two cut free proofs of the empty sequent: one is an instance of the *give up* rule and one is an instance of the *divergence* rule, with an empty context. One easily verifies that the resulting cut free proof is a *give up* (resp. a *divergence*) iff the relational interpretation of π' and π'' have a non empty intersection (resp. empty). If the interpretation is not empty we shall say that π' and π'' interact. From the point of view of the bipartite relational semantics the *give up* rule is not valid (since it introduces a sequent interpreted by a negative point) and there can be no interaction between a proof of A and a proof of $A^\perp$. We think that this is the fundamental reason which makes possible a uniform semantics where so much proofs have empty interpretations: uniformity is a restriction to possible interactions and if no interaction is possible uniformity empties things.

Remark that *give up* and *divergence* are valid rules in the others coherence like semantics we present in this paper. In particular, the property of determinism of non uniform semantics tell us that when π' and π'' interact there is only one result of this interaction (there is only one point in the intersection). This seems difficult to prove directly in the relational semantics without introducing non uniform coherence relations. Moreover the result of interaction is always in the neutral webs of the various semantics (coherence spaces, hypercoherences, multicoherences), hence, in the extended linear logic we use here, a part of the web is never visited by closed interactions. This will be the case in any extension of linear logic which admits one of these semantics.

In fact, one can imagine new para-rules such that there can be interactions on any points of the relational semantics. It has to be checked if useful, but as Curien suggested us, adding a *sum* rule like :

$$\frac{\vdash \Gamma \qquad \vdash \Gamma}{\vdash \Gamma}\,(sum)$$

interpreted by a union in the relational semantics, will certainly gives a semantics of this kind. But determinism will be lost.

6.2 Extending non uniform static semantics

Using the co-free exponential for non uniform static semantics have led to a comfortable situation where non uniform semantics are deterministic and strongly related with the uniform semantics. A (still) open question is: can the general construction Bucciarelli and Ehrhard introduced [BE01] be modified so as to directly obtain the co-free exponentials in a general way? Another related issue concerns full completeness for static semantics. Ehrhard proves a completeness theorem in an indexed linear logic framework [Ehr03]. A better understanding of the co-freeness issue in indexed linear logic may help in connecting his result with usual static semantics (hypercoherences and coherence spaces).

In an unpublished work, G. Winskel has introduced a generalization of hypercoherences. It may be interesting to adapt this generalization to a non uniform framework including the semantics we present here.

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